Worksheet on Dimension for finitely generated K-algebras

Let K be a field.

THEOREM 1: Every prime of height one in a UFD is principal.

THEOREM 2: Let R be a finitely generated over K, and let $K[x_1, \dots, x_d] \subset R$ be a Noether Normalization. Then $\dim R = d$. In particular, the polynomial ring in $K[x_1, \dots, x_d]$ has dimension d.

THEOREM 3: Every saturated chain of prime ideals from $\langle 0 \rangle$ to a maximal ideal in a domain finitely generated over K has the same length. In particular, all maximal ideals have the same height ($= \dim R$).

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(1) Using the theorems above, compute the following:
  (a) Dimension of $\mathbb{Q}[x, y, \sqrt{x^2 + y^5}]$ (a subring of $\overline{\mathbb{Q}(x, y, z)}$)
  (b) Dimension of the localization $\mathbb{Q}[x, y, \frac{1}{f}]$, where f is some non-zero polynomial in $\mathbb{Q}[x.y]$. [Hint: Use Theorem 3.]
  (c) Dimension of $\mathbb{R}[x, y, z]/\langle x^3 + yz^3 \rangle$.
  (d) Dimension of $\mathbb{C}[x, y, z]_{m_p}$ where m_p is the ideal of polynomial functions vanishing at the point $(1, 1, 1) \in \mathbb{C}^3$.
  (e) Dimension of $\mathbb{F}_{49}[x, y, z]_P$ where P is a height two prime.
  (f) The height of $\langle x_1, \dots, x_i \rangle$ in $K[x_1, \dots, x_d]$. [Hint: Use Theorem 3.]

(2) PROOF OF THEOREM 1. Let R be a UFD.
  (a) Show that every non-zero prime ideal of R contains an irreducible element.
  (b) Prove that every height one prime is principal. [Hint: Remember irreducible elements are prime!]
  (c) Prove conversely that non-zero prime ideal that is principal must have height one.

(3) PROOF OF THEOREM 2. Let $A = K[x_1, \dots, x_d] \subset R$ be a module finite extension, where A is a polynomial ring.
  (a) Show that $\dim A \geq d$ by exhibiting an explicit chain of prime ideals. [Hint: You did this before.]
  (b) Explain why, to prove Theorem 2, it suffices to show that $\dim A \leq d$.
  (c) Explain why, to prove Theorem 2, it suffices to show that for any prime ideal P_1 of height one in A, $\dim A/P_1$ has dimension $d - 1$. [Hint: What happens if $P_0 \subset P_1 \subset \cdots \subset P_{d+1}$ is a chain of primes of length more than d in A?]
  (d) For $P_1 \in \operatorname{Spec} A$ of height one, explain why after a change of variables, P_1 can be assumed to be generated by some polynomial f which is monic in x_d with coefficients in $k[x_1, \dots, x_{d-1}]$. [Hint: Use a lemma from last time.]
  (e) Use (d) to show that A/P_1 is integral over a polynomial ring in $d - 1$ variables.
  (f) Complete the proof of Theorem 2 by induction.

(4) Let $R = \operatorname{Spec} K[x, y, z]/\langle xz, yz \rangle$. Find two saturated chains in $\operatorname{Spec} R$ of different lengths. Why does this not contradict Theorem 3?

(5) Let $R = K[x, y, z, w]/\langle xz - yw \rangle$. Let P be the ideal generated by the classes of x and y.
  (a) Show that P is prime of height one. [Hint: Use Theorem 2 on the ring $K[x, y, z, w]$.]

(b) (*) Show that P not principal. [Hint: If principal, lift to a statement in the UFD $K[x, y, z, w]$ and use the grading.[1]]

(c) Why does this not violate Theorem 1? Demonstrate explicitly an instance of the failure of unique factorization in R.

(6) Dimension and Transcendence degree. Recall that if L/K is a field extension, then a **transcendence basis** for L over K is a maximal set of elements in L that are algebraically independent over K. These always exist (by Zorn's lemma) and have the same cardinality, which is called the **transcendence degree of** L/K.

Use Noether Normalization to prove that if R is a domain finitely generated over a field K, then $\dim R$ is equal to the transcendence degree of L/K where L is the fraction field of R.

(7) Proof of Theorem 3. Let $A = K[x_1, \ldots, x_d] \subset R$ be a Noether normalization for a domain R finitely generated over K.

(a) Fix any maximal ideal m in R and consider a *saturated* chain of prime ideals $\langle 0 \rangle = Q_0 \subset Q_1 \subset \cdots \subset Q_k = m$ in R. Explain why $k \leq d$ and why, to prove Theorem 3, it suffices to show that $d = k$.

(b) Explain how we know Q_1 has height one, and why to prove Theorem 3, it suffices to show that the induced chain $\langle 0 \rangle = Q_1/Q_1 \subset Q_2/Q_1 \subset \cdots \subset Q_k/Q_1$ (in the quotient ring R/Q_1) has length $d - 1$.

(c) Let $P_1 = Q_1 \cap A$. Show that P_1 is height one. [Hint: Recall (a corollary of) Going Down.]

(d) Use the same method as in the proof of Theorem 2 to show that after changing variables in A, we have module finite extensions $K[x_1, \ldots, x_{d-1}] \subset A/P_1 \subset R/Q_1$, and complete the proof of Theorem 3 by induction.

(8) Let $R = \mathbb{Z}_{\langle p \rangle}[t]$ where p is a prime integer. Observe that R is a UFD.

(a) Prove that $\langle pt - 1 \rangle$ is a maximal ideal of height one.

(b) Prove that $\langle p, t \rangle$ is a maximal ideal of height at least two.

(c) Why does this not contradict Theorem 3?

(9) Prime Avoidance Lemma.

(a) Let R be any algebra over an infinite field K, and let $I_1, \ldots, I_t$ be ideals of R. If an ideal $J \subset I_1 \cup I_2 \cup \cdots \cup I_t$, prove that $J \subset I_k$ for some index k. [Hint: This is really a statement about K-vector spaces; use induction.[2]]

(b) * For arbitrary R, assume $J \subset I_1 \cup I_2 \cup \cdots \cup I_t$ where at most two of the ideals I_k are not prime. Prove that $J \subset I_k$ for some index k.

(10) Let $\{P_n\}_{n \in \mathbb{N}} \subset \operatorname{Spec} R$ be an arbitrary collection of prime ideals in R.

(a) Show that $U = R \setminus \bigcup_{n=1}^{\infty} P_n$ is a multiplicative set.

(b) In the ring $K[x, y, z]$, let $U = K[x, y, z] \setminus (\langle x \rangle \bigcup \langle y, z \rangle)$. Find two saturated chains in $U^{-1}R$ of different lengths. [Hint: Use Prime Avoidance.]

(c) Why does this not contradict Theorem 3?

(11) Find an example of a non-Noetherian ring with finite Krull dimension.

(12) * Consider a doubly indexed set of variables $\{x_{ij} \mid i \leq j, \ i, j \in \mathbb{N}\}$. Let R be the polynomial ring they generate over $\mathbb{C}$, so $R = \mathbb{C}[x_{11}, x_{12}, x_{22}, x_{13}, x_{23}, x_{33}, \ldots]$. For each fixed j, let P_j be the

[1] If $\langle x, y \rangle = \langle f \rangle$ for some f, note that lifting f to the polynomial ring, it can not have a non-zero constant term and it must have a non-zero linear term. Using the UFD property, what is this linear term?

[2] If you need a further hint: Say $x \in J$, and wlog, $x \in I_1$. Pick $y \in J \setminus I_1$. There must be infinitely many elements of the form $x + ay$ where $a \in K \setminus \{0\}$ in some I_k, $k \geq 2$.

prime ideal generated by $\{x_{1j}, x_{2j}, \ldots, x_{jj}\}$. Let $U = R \setminus \bigcup_{j=1}^{\infty} P_j$. Prove that $U^{-1}R$ is Noetherian but has infinite Krull dimension.