Dimension and Integral Extensions

Through out, R and S denote commutative rings with identity, and K denotes a field.

DEFINITION: The (Krull) **dimension** of R is the supremum of the lengths of chains in the poset $\operatorname{Spec} R$. Explicitly,

$$\dim R := Sup\{t \mid \exists P_i \in \operatorname{Spec} R,\ s.t.\ P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_t\}$$

LYING OVER THEOREM: If $R \hookrightarrow S$ is integral, then $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective, and the fiber over any $P \in \operatorname{Spec} R$ consists of incomparable primes in $\operatorname{Spec} S$.

GOING UP COROLLARY: If $R \hookrightarrow S$ is integral, then given any chain of primes in R, say $P_0 \subset P_1 \subset \cdots \subset P_t$ and Q_0 contracting to P_0, there exists a chain of primes $Q_0 \subset Q_1 \subset \cdots \subset Q_t$ in S with the property that each Q_i contracts to P_i under the induced map of Spectra.

DIMENSION COROLLARY: If $R \hookrightarrow S$ is integral, then $\dim R = \dim S$.

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(1) For each of the rings below, find the dimension using the definition/theorem:
- (a) The field $\mathbb{C}(t)$ of meromorphic functions on the Riemann sphere.
- (b) $\mathbb{Z}$
- (c) The algebraic integers $\mathcal{A}$, i.e., the ring of all elements in $\mathbb{C}$ which satisfy an equation of integral dependence over $\mathbb{Z}$.
- (d) $\mathbb{Q}[x]$
- (e) $\mathbb{Q}[x,y]/\langle y^{17} - x^2y^3 + 5x^5\rangle$
- (f) The polynomial ring $K[x_1, \dots, x_n]$. For this one, just guess, and prove your guess is a lower bound. To show it's an upper bound is quite hard; we'll do it soon enough.
- (g) The polynomial ring $K[x_1, \dots, x_n, \dots]$
- (h) $\mathbb{Q}[x,y,z,w]/\langle w^3 - w^2x - yz, z^4 - xy\rangle$ (use (f)).

(2) Explain why $\dim W^{-1}R \leq \dim R$ and $\dim R/I \leq \dim R$.

(3) Explain why every principle ideal domain has dimension one.

(4) Prove that the ring $R \times S$ has dimension equal to the *larger* of $\dim R$ and $\dim S$.

(5) Prove that the ring R and its reduced ring R_{red} have the same dimension.

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(6) PROOFS OF THE COROLLARIES. Let $R \hookrightarrow S$ be an integral extension of rings. **Assume the Lying Over Theorem.**
- (a) Suppose that $Q_0 \in \operatorname{Spec} S$ contracts to $P_0 \in \operatorname{Spec} R$. Prove that the map $R/P_0 \hookrightarrow S/Q_0$ is also an integral extension.
- (b) Given a chain of primes $P_0 \subset P_1$ in R, prove that there exists a chain of primes $Q_0 \subset Q_1$ in S such that $P_i = Q_i \cap R$ for $i = 0, 1$. [Hint: Use (a)]
- (c) Show that the Going Up Corollary follows from the Lying Over Theorem.
- (d) Show also that the Dimension Corollary follows.

(7) Incomparable Fibers. Let $R \hookrightarrow S$ be an integral extension.
  (a) Prove that if R and S are domains, then every $u \in S$ has a non-zero multiple in R.
  (b) Show that if $Q_0 \subset Q_1$ in $\operatorname{Spec} S$ both contract to P in $\operatorname{Spec} R$, then $Q_0 = Q_1$. [Hint: Use 5a.]
  (c) Conclude that the fibers of the map $\operatorname{Spec} S \to \operatorname{Spec} R$ consist of *incomparable* primes, and that the ring $S \otimes_R R_P/PR_P$ has Krull dimension zero.

(8) Consider the extension $\mathbb{Z} \hookrightarrow \mathbb{Z}[\sqrt{2}]$ and the corresponding contraction map $\operatorname{Spec} \mathbb{Z}[\sqrt{2}] \to \operatorname{Spec} \mathbb{Z}$.
  (a) Compute the fiber over the points $\langle 0 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 7 \rangle$ of $\operatorname{Spec} \mathbb{Z}$.
  (b) For $p > 2$ a prime integer, prove that there are at most two points over $\langle p \rangle$, with exactly one if and only if 2 is not a square mod p.

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(9) The Cayley-Hamilton Theorem, revisited. Let $R \hookrightarrow S$ be an integral extension of rings. Let I be an ideal of R. **Our goal in this problem is to prove:** If $u \in S$ is contained IS, then there is an equation of integral dependence

$$u^n + r_1 u^{n-1} + \cdots + r_n = 0$$

where each $r_i \in I^i \subset R$.
  (a) For $u \in IS$, write $u = i_1 s_1 + \cdots + i_t s_t$ where $i_j \in I \subset R$ and $s_j \in S$. Explain why $R \hookrightarrow S' = R[s_1, \ldots, s_t]$ is module-finite. Now, for any set $\{a_1, \ldots, a_n\}$ of R-module generators for S' over R, explain why $IS' = Ia_1 + \cdots + Ia_n$.
  (b) Show that the $n \times n$ matrix over R representing the R-linear map $S' \to S'$ given by multiplication by u can be assumed to have all entries in I.
  (c) Prove that $u^n + r_1 u^{n-1} + \cdots + r_n = 0$ where each $r_i \in I^i \subset R$. [Hint: Cayley-Hamilton!]

(10) Easy Lemma. Let S be an arbitrary ring, and let U be a multiplicative set and J an ideal of S. Consider the two subsets, $\mathbb{V}(J)$ and $\{P \mid P \cap U = \emptyset\}$, of $\operatorname{Spec} S$. Prove that their intersection is non-empty if and only if $J \cap U = \emptyset$. [Hint: Consider the ring SU^{-1}/JSU^{-1}.]

(11) The Proof of Lying Over. Let $R \hookrightarrow S$ be an integral extension.
  (a) Let $I \subset R$ be any ideal. Prove that $IS \cap R \subset \sqrt{I}$. In particular, if I is radical, $IS \cap R = I$. [Hint: Use 8.]
  (b) Take $P \in \operatorname{Spec} R$. Show that, as subsets of S, $(R \setminus P) \cap PS = \emptyset$. [Hint: Use 9.]
  (c) Use (b) to show that for all $P \in \operatorname{Spec} R$, there exists $Q \in \operatorname{Spec} S$ such that $Q \cap R = P$.
  (d) Conclude that the Lying Over Theorem holds.

(12) Let U be a multiplicative set of a ring R that contains some non-unit. Prove that $R \to U^{-1}R$ is not an integral **extension**. [Hint: Consider the map of Spec.]

(13) Let $\mathbb{F}_2^X$ be the ring of $\mathbb{F}_2$-valued functions on an infinite set X. Prove that $\mathbb{F}_2^X$ is integeral over $\mathbb{F}_2$, and compute its dimension. [Hint: Consider the polynomial $x^2 - x = 0$.]

(14) Consider the ring extension $\mathbb{C}[y] \hookrightarrow \mathbb{C}[x, y]/\langle y - x^2 \rangle$. Let $\pi : Y \to X$ be the induced map of Spectra.
  (a) Is this ring homomorphism integral?
  (b) Explain why the closed points of Y all have the form $\langle x - a, y - b \rangle$ where (a, b) lies on the parabola $y = x^2$. Compute the contraction of $\langle x - a, y - b \rangle$ to $\operatorname{Spec} \mathbb{C}[y]$.
  (c) Compute the fiber of π over each point of X. In particular, what cardinalities are possible?
  (d) Draw a picture representing map on the subspace of closed points. Which ones are special, and in what ways?