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Worksheet on Exactness

Let R be a commutative ring with 1. Let R-Mod denote the category of R-modules.

Definition. An R-module M is **flat** if the functor from R-mod to R-mod $N \mapsto M \otimes_R N$ is exact.

Definition. An R-module P is **projective** if the functor $N \mapsto \mathrm{Hom}_R(P, N)$ is exact on R-mod.

Definition. An R-module J is **injective** if the functor $M \mapsto \mathrm{Hom}_R(M, J)$ is exact on R-mod.

Theorem. A sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact if (and only if) $0 \to (M_1)_P \to (M_2)_P \to (M_3)_P \to 0$ is exact for all $P \in \mathrm{Spec}\, R$ (equivalently, all $P \in \mathrm{maxSpec}\, R$).

Proposition. If S is a flat R-algebra and M, N are R-modules with M finitely presented,[1] then the natural homomorphism $S \otimes_R \mathrm{Hom}_R(M, N) \xrightarrow{\theta} \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N)$ is an isomorphism.

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(1) Local Properties. For any $P \in \mathrm{Spec}\, R$, let $M_P = (R \setminus P)^{-1} M$ be the localization of M at $P \in \mathrm{Spec}\, R$. Show the following:
  - (a) There is a natural R-module map $M \to M_P$ sending $m \mapsto \frac{m}{1}$, the "localization at P" map.
  - (b) Localization at P gives an *exact* functor $M \mapsto M_P$ from R-modules to R_P-modules.
  - (c) For any R-module map $M \xrightarrow{f} N$, the formation of the kernel, cokernel, and image of f commute with localization at all $P \in \mathrm{Spec}\, R$ (equivalently, $\mathrm{maxSpec}\, R$). That is, $(\ker f)_P = \ker f_P$, $(\mathrm{coker}\, f)_P = \mathrm{coker}\, f_P$, and $(\mathrm{im}\, f)_P = \mathrm{im}\, f_P$.
  - (d) For any $m \in M$, show that $\mathrm{ann}_R\, m = \{r \in R \mid \ rm = 0\}$ is an ideal of R,.
  - (e) Show $\mathrm{ann}_R\, m$ is proper if and only if $m \neq 0$.
  - (f) Show that $\frac{m}{1} \in M_P$ is nonzero if and only if $\mathrm{ann}_R\, m \subset P$.
  - (g) Show m is zero if and only if $\frac{m}{1} \in M_P$ is zero $\quad \forall P \in \mathrm{Spec}\, R$ (equivalently, $\mathrm{maxSpec}\, R$).
  - (h) Show $M = 0$ if and only if $M_P = 0 \ \ \forall P \in \mathrm{Spec}\, R$ (equivalently, $\mathrm{maxSpec}\, R$).
  - (i) For any R-module map $f : M \to N$, f is injective (respectively surjective) if and only if f_P is injective (respectively, surjective) for all $P \in \mathrm{Spec}\, R$ (equivalently, $\mathrm{maxSpec}\, R$).
  - (j) Two submodules N_1, N_2 of M satisfy $N_1 \subset N_2$ if and only if $(N_1)_P \subset (N_2)_P$ in M_P for all $P \in \mathrm{Spec}\, R$ (equivalently, $\mathrm{maxSpec}\, R$). [Hint: $N_1 \subset N_2$ if and only if $(N_1 + N_2)/N_2 = 0$.]
  - (k) Prove the theorem.

(2) Fix any R-module M. Prove the following exactness properties of the following functors from R-mod to R-mod.
  - (a) Show that the covariant functor $N \mapsto \mathrm{Hom}_R(M, N)$ is left exact.
  - (b) Show that the contravariant functor $M \mapsto \mathrm{Hom}_R(M, N)$ is left exact.

(3) Projective Modules.
  - (a) Show that P is projective if and only if for every surjection $g : M_2 \twoheadrightarrow M_3$ and morphism $\phi : P \to M_3$, there exists $\tilde{\phi} : P \to M_2$ such that $\phi = g \circ \tilde{\phi}$.
  - (b) Show that every free R-module is projective. [Use (a) and the Universal property of free modules.]
  - (c) Show that if P is projective, then every surjection $M \twoheadrightarrow P$ splits in the category of R-modules. [Hint: Use (a).]

[1] Meaning there is a short exact sequence $G \to F \to M \to 0$ where F and G are free and finite rank.

(d) Show that if P is projective, then P is a direct summand of a free module. [Hint: Map a free module surjectively onto P.]
(e) Conversely, suppose that P be an R-module with the property that there exists an R-module Q such that $P \oplus Q$ is free. Show that P is projective. [One straightforward way uses (a) and (b).]

(4) Projectivity can be tested locally for finitely presented modules. *Assume the Proposition above on commutation of Hom with flat base change.*
(a) Prove that a finitely presented R-module Q is projective if and only if Q_P is a projective R_P-module for all $P \in \operatorname{Spec} R$ (or maxSpec R).
(b) Prove that if R is Noetherian, then this statement holds also for finitely generated R-modules. [Hint: Map a finitely generated free module onto M and consider the kernel.]

(5) The Proof that Hom Commutes with Flat Base Change. Let S be an R-algebra. Fix an R-module N.
(a) For any R-module M, describe a natural S-module map

$$S \otimes \operatorname{Hom}_R(M, N) \xrightarrow{\theta_M} \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N).$$

[Hint: First find an R-linear map.]
(b) Fix N. Show that $\theta = \theta_M$ defines a natural transformation from the functor

$$S \otimes_R \operatorname{Hom}_R(-, N) \quad \text{to} \quad \operatorname{Hom}_S(S \otimes_R -, S \otimes_R N).$$

This means that, given an R-module map $M_1 \xrightarrow{g} M_2$, an appropriate diagram involving θ_{M_1} and θ_{M_2} commutes.
(c) Show that θ_R is an isomorphism.
(d) Show that $\theta_{M_1 \oplus M_2}$ can be identified with $\theta_{M_1} \oplus \theta_{M_2}$. In particular, show that θ_F is an isomorphism if F is a finitely generated free module.
(e) Prove the Proposition. [Hint: Show the map in (a) is an isomorphism by using (2b) and right exactness of tensor to construct a commuting diagram from a presentation of M. The desired isomorphism will come from the kernels of isomorphic maps.]

(6) Injective Modules.
(a) Show that J is injective if and only if for every injection $g : M_1 \hookrightarrow M_2$ and morphism $\phi : M_1 \to J$, there exists $\tilde{\phi} : M_2 \to J$ such that $\tilde{\phi} = g \circ \phi$.
(b) Show if J is injective, then every inclusion $J \hookrightarrow M$ splits in R-mod. [Hint: Use (a).]
(c) * Prove the converse of (b).

(7) Baer's Criterion. In this problem, we prove: *An R-module E is injective if and only if for every ideal I of R, any map $I \to E$ extends to a map $R \to E$.*
(a) Fix an R module E. One direction is trivial (why?). So assume that for any ideal I of R, any map $I \to E$ extends to a map $R \to E$.
(b) Fix an inclusion of R modules $M \subset N$ and map $\phi : M \to E$. Explain why we can find a maximal extension $N' \subset N$ of M to which ϕ extends. Explain why we are done if $N' = N$.
(c) Pick an $n \in N \setminus N'$. Show that the set $\mathfrak{a} := \{r \in R \mid rn \in N'\}$ is an ideal of R.
(d) Consider the composition $\mathfrak{a} \xrightarrow{\times n} N \xrightarrow{\phi} E$. Prove it extends to an R-module map $\psi : R \to E$.
(e) Show that $N' + nR \to E$ sending $n' + rn \mapsto \phi(n') + \psi(r)$ is a well-defined R-module map extending the given map ϕ. Complete the proof of Baer's Criterion. [Hint: Cf (b).]

(8) If R is a domain, prove that its fraction field is an injective module.