Going Down and Integral Extensions

Through out, R and S denote commutative rings with identity, and K denotes a field.

DEFINITION: A domain R is **normal** if every element of its fraction field that is integral over R is actually in R.

PROPOSITION: Every UFD is normal.

DEFINITION: The **height** of a prime ideal P is $\sup\{d \mid \exists P_d = P \supset P_{d-1} \supset \cdots \supset P_1 \supset P_0\}$.

GOING DOWN THEOREM: Let $R \hookrightarrow S$ be an integral extension with R a normal domain. Assume S is a torsion-free as an R-module. Then given any chain of primes $P_n \supset P_{n-1} \supset \cdots \supset P_0$ in $\operatorname{Spec} R$, and a prime $Q_n \in \operatorname{Spec} S$ such that $Q_n \cap R = P_n$, we can lift the entire chain to a chain $Q_n \supset Q_{n-1} \supset \cdots \supset Q_0$ in $\operatorname{Spec} S$ such that $Q_i \cap R = P_i$.

COROLLARY: Let $R \subset S$ be an integral extension of domains with R normal. For all $Q \in \operatorname{Spec} S$, we have height Q = height P, where $P = Q \cap R$.

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(1) Prove that for any $P \in \operatorname{Spec} R$, the height of P equals the dimension of R_P.

(2) Prove the Proposition. [Hint: Write an integral element of the fraction field in lowest terms.]

(3) Which of the following are normal?
  (a) $\mathbb{Z}$
  (b) $\mathbb{Z}[x, y, z]$
  (c) The algebraic integers $\mathcal{A}$, i.e., the ring of all elements in $\mathbb{C}$ which satisfy an equation of integral dependence over $\mathbb{Z}$. [Hint: remember that a composition of integral extensions is integral.]
  (d) $\mathbb{C}[[x, y, z]]$.
  (e) $\mathbb{Q}[x^2, x^3]$

(4) Let $R \hookrightarrow S$ be an integral extension. Suppose that $Q \in \operatorname{Spec} S$ lies over $P \in \operatorname{Spec} R$.
  (a) Prove that height $P \geq$ height Q. [Hint: Use Lying Over.]
  (b) Assuming the **Going Down** Theorem, prove its COROLLARY.

(5) Let $R = K[x]$ and $S = K[x, y]/\langle y^2 - y, xy\rangle$. Let $Q \subset S$ be the ideal $\langle y - 1\rangle$.
  (a) Show that $R \subset S$ is an integral extension.
  (b) Show that $x \in Q$.
  (c) Prove Q is prime in S and that S_Q is a field.
  (d) Show that $Q \cap R = \langle x\rangle$.
  (e) Compare the height of Q and the height of its contraction to R.
  (f) Why is this not contradicting Going Down?

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(6) Let $R \subset S$ be an extension of rings, and assume that the map splits in the category of R-modules. This means that there exists an R-module map $S \to R$ which restricts to the identity on R.
  (a) Show that if S is normal, then R normal.

(b) Consider the ring $R = k[x^2, xy, y^2]$ generated by the degree two monomials in the polynomial ring $k[x, y]$. Show that R is normal, but not a UFD.

(7) Let $R = K[s(1-s), t, st] \subset K[s, t] = S$.
   (a) Show that this is an integral extension of domains. [Hint: $f(x) = x^2 - x + s(1-s)$.]
   (b) Show that $Q = \langle 1-s, t\rangle \in \operatorname{Spec} S$ contracts to $P = \langle s(1-s), t, st\rangle$, which is maximal.
   (c) Show that $\langle s\rangle \in \operatorname{Spec} S$ contracts to $P_0 = \langle s(1-s), st\rangle$. [Hint: what elements of R are not in P_0?]
   (d) Show that no prime Q_0 contained in Q lies over P_0.
   (e) Why doesn't this contradict Going Down?

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(8) We will prove the following **Lemma:** *Let R be a normal domain with fraction field K, and let $\overline{K}$ be the algebraic closure of K. If $s \in \overline{K}$ is integral over R, then the minimal polynomial for s over K is in $R[x]$.*
   (a) Fix $s \in \overline{K}$ integral over R. Let $h(x) \in R[x]$ be a monic equation of integral dependence over R and let $f(x)$ be the minimal polynomial of s over K. Show that f divides h in the ring $K[x]$.
   (b) Show that every root of f is integral over R. [Hint: Use (a).]
   (c) Show that the coefficients of f are integral over R. [Hint: elementary symmetric functions.]
   (d) Using the fact that R is normal, conclude that $f(x) \in R[x]$.

(9) Proof of Going Down: Main Case. Let $R \hookrightarrow S$ be an integral extension of domains with R normal. Let K be the fraction field of R.
   (a) Reduce the proof to the following: If $Q \in \operatorname{Spec} S$ lies over $P \in \operatorname{Spec} R$, and $P_0 \in \operatorname{Spec} R$ satisfies $P_0 \subset P$, then there exists $Q_0 \in \operatorname{Spec} S$ such that $Q_0 \subset Q$ and $Q_0 \cap R = P_0$.
   (b) (*) Consider the subset $(R \setminus P_0)(S \setminus Q) \subset S$. Prove that this is a multiplicative set that does not meet $P_0 S$.[1]
   (c) Using (b), show that there is a prime Q_0 of S which contains $P_0 S$ and is contained in Q.
   (d) Conclude that (a) holds, and hence the Going Down Theorem is proved in this case.

(10) Proof of Going Down: Full Statement. Let $R \hookrightarrow S$ be an integral extension with R a normal domain. Assume S is a torsion-free as an R-module.
   (a) Observe that the reduction in 8(a) is still valid.
   (b) Show that $W = (R \setminus \{0_R\})(S \setminus Q)$ is a multiplicative set in S such that $S \to W^{-1}S$ is not the zero ring.
   (c) Show that there exists $q \in \operatorname{Spec} S$ such that $q \subset Q$ and $q \cap R = 0$. [Hint: Start by picking any prime in $W^{-1}S$ and considering the corresponding point in $\operatorname{Spec} S$.]
   (d) For q as in (b), show that $R \hookrightarrow S/q$ is an integral extension, and that to prove Going Down, it suffices to prove (a) for $R \subset S/q$. This completes the proof of Going Down, using 9.

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[1] [Hint: Say $r \in R \setminus P_0$ and $s \in S \setminus Q$ are such that $rs \in P_0 S$. Recall from the Cayley Hamilton Trick (Problem 9 on the last worksheet), the monic polynomial $g(x)$ of rs can be assumed to have all non-leading coefficients in $P_0 \subset R$. Set $h(x) = g(rx)$ and show that h is divisible in $R[x]$ by the minimal polynomial $f(x)$ of s over K (which by (8) is in $R[x]$). Now show that all non-leading terms of f are in P_0 and derive a contradiction.]