Worksheet on Graded Rings

Let $(A, +)$ be a commutative monoid— a set with an associative binary operation and identity element 0_A (but unlike a group, not every element need have an inverse). The main examples are $\mathbb{N}$ and $\mathbb{N}^d$.

DEFINITION. A commutative ring R is **graded by** the monoid A if $(R, +)$ decomposes as

$$R = \bigoplus_{a \in A} R_a$$

in the category of abelian groups, and for all $a, b \in A$, we have $R_a \cdot R_b \subset R_{a+b}$.

DEFINITION. Let R be an A-graded ring. The non-zero elements of R_a are said to be **homogeneous of degree** a. An ideal $I \subset R$ is said to be **homogeneous** if I can be generated by homogeneous elements.

DEFINITION. A morphism in the category of A-graded rings is a ring homomorphism $\phi : R \to S$ such that for all $a \in A$, $\phi(R_a) \subset S_a$.

DEFINITION. Let R be an A-graded ring. We say that an R-module M is A-graded if M decomposes in the category of abelian groups as

$$M = \bigoplus_{a \in A} M_a$$

such that $R_a \cdot M_b \subset M_{a+b}$ for all $a, b \in A$. A morphism in the category of A-graded R-modules is a *degree-preserving* R-module map.

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(1) Let R be an A-graded ring.
  (a) Show that 1_R is homogeneous of degree $0 \in A$. [Hint: For $r \in R_b$, consider $1 \cdot r$. Use the fact that the multiplicative identity in a ring is unique.]
  (b) Show that R_0 is a subring of R, so that R is an R_0-algebra.
  (c) Show that each homogeneous component R_a has a natural R_0-module structure.

(2) Let $R = K[x, y]$. Verify that each of the following uniquely defines a **graded ring structure** on R, and describe the homogeneous components R_a for arbitrary $a \in A$. Show that none are isomorphic as graded rings.
  (a) THE STANDARD GRADING BY $\mathbb{N}$. Here x, y both have degree $1 \in \mathbb{N}$.
  (b) A NON-STANDARD GRADING BY $\mathbb{N}$. Define $\deg x = 2$ and $\deg y = 3$.
  (c) ANOTHER NON-STANDARD GRADING BY $\mathbb{N}$. Define $\deg x = 0$ and $\deg y = 1$.
  (d) THE STANDARD MONOMIAL GRADING BY $\mathbb{N}^2$. Define $\deg x = (1, 0)$ and $\deg y = (0, 1)$.
  (e) A NON-STANDARD GRADING BY $\mathbb{N}^2$. Define $\deg x = (1, 0)$ and $\deg y = (0, 2)$.

(3) Let $R = K[x, y]/\langle y^7 - x^5(x+1)\rangle$. Show that R has a natural $\mathbb{Z}/7\mathbb{Z}$-grading whose zero-th graded piece is $K[x]$. [Hint: Note that R is a free $K[x]$-module of rank 7 with basis classes of (certain) powers y.]

(4) Let R be an A-graded ring, and let $I \subset R$ be any ideal. Show that the following are equivalent.
  (a) I can be generated by homogeneous elements of R.
  (b) As an abelian group, $I = \oplus_{a \in A} I \cap R_a$. [Hint: Note that an arbitrary R-linear combination of the homogeneous generators for I is a sum of homogeneous elements of I.]

(c) I is an A-graded R-submodule of R—that is, the inclusion $I \hookrightarrow R$ is a map of graded R-modules.
(d) Whenever $f = \sum_{a \in A} f_a$ is in I, where the f_a are homogeneous of degree a, it follows that all components $f_a \in I$.

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(5) Show that if $I \subset R$ is a homogenous ideal, then there is a natural A-grading on the quotient ring R/I such that the quotient map $R \to R/I$ is a morphism in the category of A-graded rings.

(6) All of the following quotients of $K[x, y, z, w]$ have $\mathbb{N}$-gradings. Describe one non-trivial grading for each.
(a) $K[x, y, z, w]/\langle xy - zw \rangle$
(b) $K[x, y, z, w]/\langle x^2 + y^3 + z^5 + w^7 \rangle$
(c) $K[x, y, z, w]/\langle xy - z^2, w^4 - xy \rangle$

(7) For the rings in 6(a) and 6(c), find
(a) An $\mathbb{N}^2$-grading that has zero-th graded piece K.
(b) An $\mathbb{N}$-grading that has zero-th graded piece $K[x]$.

(8) Let K be an algebraically closed field, and let S be the polynomial ring $K[x, y]$ with its standard grading.
(a) Show that every homogeneous polynomial in S factors into homogeneous polynomials of degree one. [Hint: For $f(x, y)$ homogeneous of degree d, show $\frac{f(x,y)}{y^d} = f(\frac{x}{y}, 1)$.]
(b) Describe the homogeneous prime ideals in S. There are three types. Describe their heights.
(c) Define a bijection between the set of non-zero, non-maximal homogenous prime ideals in S and the set of one dimensional vector subspaces of K^2.

(9) Let S be an $\mathbb{N}$-graded ring.
(a) Show that $S_+ = \bigoplus_{n \in \mathbb{N}} S_n$ is an ideal of S, called the **irrelevant ideal.**
(b) Show that a homogeneous ideal $I \subset S$ is prime if and only if for all *homogeneous* $f, g \in S$, if $fg \in I$ then either f or g is in I. [Hint: If $fg \in I$, use induction on the total number of non-zero terms in f and g together.]
(c) Show that S_+ is prime if and only if S_0 is a domain, and S_+ is maximal if and only if S_0 is a field.

(10) Define Proj S to be the subset of Spec S consisting of *homogeneous prime ideals* in the open set Spec $S \setminus V(S_+)$. Give Proj S the subspace Zariski topology inherited from Spec S.
(a) Show that every closed set of Proj S has the form $\mathbb{V}(I) = \{P \in \text{Proj} S \mid I \subset P\}$ for some homogeneous ideal I. [Hint: Use 4(d).]
(b) Let $f \in S$ be homogeneous in S. Show that the set $D_+(f) = \{P \in \text{Proj } S \mid f \notin P\}$ is open.
(c) Show that Proj S has a basis by sets of the form $D_+(f)$ where f is homogeneous.

(11) Let $S = \mathbb{C}[x, y]$ with its standard grading. Consider Proj S with its Zariski topology described above, and let $\mathcal{X} \subset$ Proj S be the subspace of **closed points.**
(a) Show that has a dense point—that is, a point η that is in every non-empty open set.
(b) Show that all the other points in Proj S are closed—that is, $\mathcal{X} =$ Proj $S \setminus \{\eta\}$.
(c) Show that the Zariski topology on $\mathcal{X}$ is the finite-complement topology.
(d) Show that $\mathcal{X}$ can be interpreted as the set of one-dimensional sub-vector spaces of $\mathbb{C}^2$, with the finite complement topology.