Worksheet on I-adic Topology and Completion

Let R be a ring and $I \subset R$ an ideal. Let $R^{\mathbb{N}}$ denote the ring of all sequences $(r_n)_{n\in\mathbb{N}}$ in R.

Definition. We say a sequence in $R^{\mathbb{N}}$ is **Cauchy for the I-adic topology** if for all $t \in \mathbb{N}$, there exists $d \in \mathbb{N}$ such that whenever $n, m \geq d$, we have $r_n - r_m \in I^t$.

(1) Show that the subset $\mathfrak{C}^I(R)$ of Cauchy sequences in $R^{\mathbb{N}}$ is a subring.
(2) Show that the map $R \to \mathfrak{C}_I(R)$ sending r to the constant sequence (r) is a ring homomorphism, making $\mathfrak{C}_I(R)$ into an R-algebra.
(3) Let $\mathfrak{C}_I^0(R)$ be the set of sequences $(r_i)_{i\in\mathbb{N}}$ that converge to zero, meaning that for all n there exists m such that for all $i \geq m$, $r_i \in I^n$. Prove that $\mathfrak{C}_I^0(R)$ is an ideal of $\mathfrak{C}_I(R)$.
(4) Observe that a subsequence of a Cauchy sequence is Cauchy, and differs from the original by a sequence converging to zero.

Definition. The **completion of R in the I-adic topology** is the ring $\widehat{R}^I = \mathfrak{C}_I(R)/\mathfrak{C}_I^0(R)$.

(5) The elements of $\widehat{R}^I$ are equivalence classes of Cauchy sequences that differ by Cauchy sequences converging to zero. We call such an equivalence class a **limit of a Cauchy sequence** in R. Compare this to the construction of the real numbers as the collection of limits of Cauchy sequences of rational numbers.
(6) Discuss a natural map $R \to \widehat{R}^I$. What is its kernel?
(7) Suppose (R, m) is Noetherian and local. Prove that $R \to \widehat{R}^m$ is injective.

(8) Let $R = \mathbb{Z}$ and $I = \langle p \rangle$. Show that

$$(\sum_{i=0}^{n} p^i)_{n\in\mathbb{N}} = (1,\ 1+p,\ 1+p+p^2,\ 1+p+p^2+p^3, \dots)$$

is a Cauchy sequence for the p-adic topology on $\mathbb{Z}$. Show also that it represents the inverse of $1-p$ in the completion $\widehat{\mathbb{Z}}^p$ (also denoted $\widehat{\mathbb{Z}}_p$.)

(9) Let $R = K[x, y]$ and $I = \langle x, y \rangle$.
  (a) Show that

$$(\sum_{i=0}^{n} r_i x^i y^{i+2})_{n\in\mathbb{N}} = (r_0 y^2,\ r_0 y^2 + r_1 x y^3,\ r_0 y^2 + r_1 x y^3 + r_2 x^2 y^4,\ \dots)$$

  is a Cauchy sequence in the I-adic topology.
  (b) Write a Cauchy sequence different from the one in (a) but which has the same limit (that is, represents the same element of $\widehat{R}^I$, or equivalently, differs by a sequence converging to zero).
  (c) Describe a natural ring map[1] $K[[x, y]] \to \widehat{R}^I$ taking a power series $\sum_{i=0}^{\infty} r_{ij} x^i y^j$ to an equivalence class of Cauchy sequences. Can you show it's injective?

[1]think: successive truncations

(10) Explain how to think of an element of $\widehat{R}^I$ (non-uniquely) as a "formal power series" $\sum_{n=0}^{\infty} x_n$ where $x_n \in I^n$. This might help you prove surjectivity in (9c).

THEOREM. Let $S = K[x_1, \ldots, x_n]$ and $I = \langle x_1, \ldots, x_n \rangle$. Then $\widehat{S}^I = K[[x_1, \ldots, x_n]]$. Furthermore, let $g_1, \ldots, g_t \in K[x_1, \ldots, x_n] \subset K[[x_1, \ldots, x_n]]$. Then

$$\text{for } R = \frac{K[x_1, \ldots, x_n]}{\langle g_1, \ldots, g_t \rangle} \quad \text{we have} \quad \widehat{R}^I = \frac{K[[x_1, \ldots, x_n]]}{\langle g_1, \ldots, g_t \rangle}.$$

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(11) Assume the Theorem. Let $R = \mathbb{R}[x, y]/\langle y^2 - x^2 - x^3 \rangle$. Let $m = \langle x, y \rangle$.
  (a) Prove that R is a domain.
  (b) Show that $\widehat{R}^m$ is not a domain. [Hint: find a power series representing $\sqrt{1+x}$.]
  (c) Draw a picture of the curve in $\mathbb{R}^2$ with coordinate ring R.
  (d) There is a sense in which $\operatorname{Spec} \widehat{R}^m$ can be considered a very small neighborhood around $m \in \operatorname{Spec} R$. How is this reflected in your picture?

(12) Let $(r_n)_{n \in \mathbb{N}} \in \mathfrak{C}_I(R)$ be a Cauchy sequence. Fix $t \in \mathbb{N}$.
  (a) Explain why the sequence of residues $r_n \mod I^t$ is eventually constant as $n \to \infty$.
  (b) Use (a) to show there is a surjective ring homomorphism $\mathfrak{C}_I \twoheadrightarrow R/I^t$.
  (c) Use (b) to show there is a surjective ring homomorphism $\widehat{R}^I \twoheadrightarrow R/I^t$.
  (d) Show that for any $t' \leq t$, the map for t' is the composition $\widehat{R}^I \twoheadrightarrow R/I^t \twoheadrightarrow R/I^{t'}$.

DEFINITION. A **directed set** is a partially ordered set $(\Lambda, \leq)$ with the property that for any $\lambda_1, \lambda_2 \in \Lambda$, there exists $\mu \in \Lambda$ with $\lambda_i \leq \mu$. View $(\Lambda, \leq)$ as a category whose objects are $\lambda \in \Lambda$, with exactly one morphism from λ to μ if $\lambda \leq \mu$ (and none otherwise).

DEFINITION. A **direct limit system** in a category $\mathcal{C}$ is collection of objects X_λ in $\mathcal{C}$ indexed by Λ, together with morphisms $f_{\lambda\mu} : X_\lambda \to X_\mu$ whenever $\lambda \leq \mu$. These morphisms satisfy $f_{\lambda\lambda}$ is the identity on X_λ and, whenever $\lambda \leq \mu \leq \nu$, we have $f_{\mu\nu} \circ f_{\lambda\mu} = f_{\lambda\nu}$. Briefly put, a direct limit system in $\mathcal{C}$ is a covariant functor from $(\Lambda, \leq)$ to $\mathcal{C}$. An **inverse limit system** in $\mathcal{C}$ is direct limit system in $\mathcal{C}^{op}$, or equivalently, a contravariant function from $(\Lambda, \leq)$ to $\mathcal{C}$.

(13) Write out the definition of inverse limit system (in the category of rings, say) explicitly.

(14) Fix a topological space X.
  (a) Show that the collection Λ of all open sets of X is a directed set, with the partial order being inclusion.
  (b) Let $\mathcal{O}_X(U)$ be ring of continuous real-valued functions on $U \in \Lambda$. Show that restriction induces an inverse limit system indexed by Λ in the category of commutative rings.

(15) Fix a ring R and and ideal I.
  (a) Show that $\mathbb{N}$ is a directed set.
  (b) Show that the natural quotient maps $R/I^n \to R/I^m$ (whenever $n \geq m$) form an inverse limit system.

Definition. Fix a direct limit system in a category $\mathcal{C}$. An object X of $\mathcal{C}$ is its **direct limit** if there are morphisms $g_\lambda : X_\lambda \to X$ commuting with all the maps in the limit system such that the following universal property is satisfied: for any object Y with the property that $X_\lambda \to Y$ compatibly the maps in the limit system, there is a unique morphism $X \to Y$ making all diagrams commute. An **inverse limit** of an inverse limit in $\mathcal{C}$ is the direct limit in $\mathcal{C}^{op}$.

(16) Write out the definition of inverse limit (in the category of rings, say) explicitly.
