

## Worksheet on Integral Extensions

Fix a commutative ring  $R$  with identity  $1_R$ .

**DEFINITION:** A (commutative)  $R$ -**algebra** is a ring<sup>1</sup>  $A$  together with a ring homomorphism  $R \rightarrow A$ .

**DEFINITION:** An  $R$ -algebra  $A$  is **module-finite** if  $A$  is finitely generated **as an  $R$ -module**.

**DEFINITION:** Given an  $R$ -algebra  $A$ , we say that  $a \in A$  is **integral over  $R$**  if  $a$  satisfies a **monic** polynomial  $f(x) \in R[X]$ . The  $R$  algebra  $A$  is **integral** if *every*  $a \in A$  is integral over  $R$ .

**PROPOSITION:** If some generating set for an  $R$  algebra  $A$  consists of integral elements, then every element of  $A$  is integral over  $R$ —that is,  $A$  is integral over  $R$ .

**THEOREM:** An  $R$ -algebra  $A$  is module-finite if and only if it is both finitely generated (as an  $R$ -algebra) and integral.

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- (1) For each algebra below, discuss whether or not the algebra has the following properties: finitely generated, module finite, and/or integral.
    - (a) The  $\mathbb{Z}$ -algebra of Gaussian integers  $\mathbb{Z}[i]$ .
    - (b) The  $\mathbb{Q}$ -algebra  $\mathbb{C}$ .
    - (c) The  $\mathbb{R}$ -algebra  $\mathbb{C}$ .
    - (d) The  $\mathbb{Z}$ -algebra  $\mathbb{Z}[\{\sqrt{n} \mid n \in \mathbb{N}\}]$ .
    - (e) The quotient  $\mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]/\langle x^2 - y^{17} \rangle$
    - (f) The inclusion  $\mathbb{C}[x] \subset \mathbb{C}[x, y]$
    - (g) The localization  $\mathbb{Q}[x] \rightarrow \mathbb{Q}[x, \frac{1}{x}]$
    - (h) The polynomial ring  $\overline{\mathbb{F}_2}[x_1, \dots, x_n, \dots]$  in infinity many variables over the algebraic closure of  $\mathbb{F}_2$ .
    - (i) The power series ring  $\mathbb{C}(t)[[x]]$  over the field  $\mathbb{C}(t)$ .
  - (2) Prove that every module-finite  $R$ -algebra  $A$  is finitely generated as an  $R$ -algebra. That is, show that “module-finite” implies “algebra-finite”. Is the converse true?
  - (3) Let  $R$  be Noetherian. Prove that every finitely generated  $R$ -algebra is a Noetherian ring.

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- (4) Suppose that  $A \rightarrow B \rightarrow C$  are ring homomorphisms.
    - (a) Prove that if  $B$  is algebra finite over  $A$  and  $C$  is algebra finite over  $B$ , then  $C$  is a finitely generated  $A$ -algebra.
    - (b) Prove that if  $C$  is a finitely generated  $A$ -algebra, then also  $C$  is a finitely generated  $B$ -algebra.
    - (c) Give an example to show that, even when the maps  $A \rightarrow B \rightarrow C$  are injective, finite generation of  $C$  over  $A$  does not imply that finite generation of  $B$  over  $A$ . [Hint: Consider the  $K$ -sub-algebra  $B = K[x, xy, xy^2, xy^3, \dots]$  of  $K[x, y]$ .]

### (5) INTEGRAL EXTENSIONS

- (a) Show that a composition of module finite ring maps is module finite.

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<sup>1</sup>As usual in 614, we assume that  $A$  is commutative with identity and ring maps preserve the identity.

- (b) Let  $R \rightarrow S$  be a ring map. Prove that if  $s \in S$  is integral over  $R$ , then  $R[s]$  is module finite over  $R$ .
- (c) Prove that if an  $R$ -algebra  $A$  can be generated by finitely many integral elements  $\{a_1, \dots, a_n\}$ , then  $A$  is module finite over  $R$ .
- (6) **THE DETERMINANT TRICK.** Let  $B$  be an  $n \times n$  matrix with entries in  $R$ . We will now prove the following useful matrix equation (which is well-known when  $R$  is a field):

$$B \times \text{adj}(B) = \det B \times I_{n \times n}$$

where  $\text{adj}(B)$  denotes the adjugate matrix of  $B$ —that is, the transpose of the  $n \times n$  matrix whose  $ij$ -th entry is  $(-1)^{i+j}$  times the determinant of the  $(n-1) \times (n-1)$  matrix obtained by deleting row  $i$  and column  $j$  from  $B$ .

- (a) Observe that an arbitrary ring has a canonical  $\mathbb{Z}$ -algebra structure. Use this to show that  $R$  is isomorphic to a quotient of a polynomial ring  $\mathbb{Z}[\mathbf{X}]$  over  $\mathbb{Z}$  (possibly in infinitely many variables).
- (b) Show that to prove the matrix equation for  $R$ , it suffices to prove the formula for  $\mathbb{Z}[\mathbf{X}]$ . [Hint: use the surjection  $\mathbb{Z}[\mathbf{X}] \rightarrow R$  to lift each entry  $b_{ij}$  of  $B$  to  $a_{ij}$  in  $\mathbb{Z}[\mathbf{X}]$ .]
- (c) Show that to prove the matrix equation over  $\mathbb{Z}[\mathbf{X}]$ , it suffices to prove it over its fraction field. [Hint: Don't forget that two matrices are equal means that their corresponding entries are equal.]
- (d) Viola! Conclude the matrix equation above from the well-known corresponding linear algebra fact (over fields). Be sure to check the field case too, if it's not familiar to you. It's easy, if you know about computing determinants by "expanding along a row or column".
- (7) **MODULE FINITE IMPLIES INTEGRAL.** Let  $R \rightarrow A$  be a module finite ring homomorphism. Our goal is to prove that it is integral. [You have considered the converse in Problem 5.]

**Our strategy:** Pick any  $a \in A$ . The idea is a Cayley-Hamilton type result: the  $R$ -linear map  $A \rightarrow A$  given by multiplication by  $a$  satisfies its own characteristic polynomial over  $R$ .

- (a) Take any  $a \in A$ . Prove that the map  $A \rightarrow A$  sending  $b \mapsto ab$  is  $R$ -linear.
- (b) Fix  $R$ -module generators  $a_1, a_2, \dots, a_n$  for  $A$ . **Without loss of generality, assume  $a_1 = 1$ .** Explain how to think of any element of  $A$  as a column vector with entries in  $R$  and any  $R$ -linear map  $A \rightarrow A$  as given by (left) multiplication by some  $n \times n$  matrix with entries in  $R$ .
- (c) For  $a \in A$ , let  $A$  be the matrix of the map "multiplication by  $a$ ". Prove the matrix equation in  $A$

$$a I_{n \times n} v = A v, ,$$

where  $v$  is the column vector  $[1 \ a_2 \ \dots \ a_n]^{tr}$  made up of our  $R$ -module generators of  $A$ .

- (d) Let  $B = (A - a I_{n \times n})$ . Show that  $\det B = 0$ . [Hint: Multiply both sides of the matrix equation  $Bv = 0$  by  $\text{adj}(B)$ . **If you are thinking  $A$  is a diagonal matrix you are making a big mistake!**]
- (e) Prove that  $a \in A$  satisfies a monic polynomial with coefficients in  $R$ .
- (f) Conclude the that Theorem in page 1 is proved!

- (8) **INTEGRAL CLOSURE.** Let  $R \hookrightarrow S$  be a ring extension. The **integral closure of  $R$  in  $S$**  is the set of all elements of  $S$  that are integral over  $R$ .
- (a) Prove that the integral closure of  $R$  in  $S$  is a **ring**, intermediate between  $R$  and  $S$ .
- (b) Compute the integral closure of the ring  $\mathbb{Q}[x^2, x^3]$  in  $\mathbb{Q}[x]$ .
- (c) \* If  $R$  is a domain and  $S$  is its fraction field, then the integral closure of  $R$  in  $S$  is also called the **normalization** of  $R$ . Compute the normalization of  $\mathbb{Q}[x^2, x^3]$
- (d) \* Compute the normalization of  $\mathbb{Z}[\sqrt{-5}]$ .