# Worksheet on Inverse Limits and Completion

Let $R$ be a ring, $I \subset R$ an ideal, and $M$ and $R$-module.

Definition. A sequence in $(x_n) \in M^{\mathbb{N}}$ is **Cauchy for the $I$-adic topology** if for all $t \in \mathbb{N}$, there exists $d \in \mathbb{N}$ such that whenever $n, m \geq d$, we have $x_n - x_m \in I^t M$.
It converges to zero if for all $t \in \mathbb{N}$, there exists $d \in \mathbb{N}$ such that $x_n \in I^t M$ whenever $n \geq d$.

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(1) Completion of Modules. Let $\mathfrak{C}_I(M)$ denote the set of Cauchy sequences in $M$.
  (a) Prove that $\mathfrak{C}_I(M)$ is a module over the ring $\mathfrak{C}_I(R)$.
  (b) Prove the set $\mathfrak{C}_I^0(M)$ of sequences converging to zero forms a $\mathfrak{C}_I(R)$-submodule $\mathfrak{C}_I(M)$.
  (c) Prove the abelian group $\mathfrak{C}_I(M)/\mathfrak{C}_I^0(M)$ has the structure of an $\widehat{R}^I$-module. We denote this module by $\widehat{M}^I$, and call it the **completion of $M$ along $I$.**

(2) Inverse Limits. Consider the **inverse limit system** maps of sets

$$X_1 \xleftarrow{\nu_{12}} X_2 \xleftarrow{\nu_{23}} X_3 \xleftarrow{\nu_{34}} X_4 \xleftarrow{\nu_{45}} \cdots$$

and let $\nu_{ij}$ be the resuling composition from $X_j$ to $X_i$. Define the **inverse limit** $\varprojlim X_n$ to be the subset of the direct product $\Pi_{i=1}^{\infty} X_i$ consisting of all sequences

$$(x_1, x_2, x_3, \dots)$$

such that $\nu_{ij}(x_j) = x_i$ for all $i, j$.
  (a) Prove that if the $X_i$ are abelian groups and the $\nu_{ij}$ are group homomorphisms, then $\varprojlim X_n$ has a naturally induced structure of an abelian group.
  (b) Prove there are natural group homomorphisms $\varprojlim X_n \xrightarrow{\pi_n} X_n$ for all $n$ such that $\pi_m = \nu_{mn}\pi_n$ for all $n \geq m$.
  (c) Verify that (a) and (b) hold also in the category of rings, as well as the category of $R$-modules.
  (d) Assume that (in the setting of (a)) $G$ is an abelian group such that
   - There is a group homomorphism $G \xrightarrow{\phi_n} X_n$ for all $n$, and
   - $\phi_m = \nu_{mn} \circ \phi_n$ for all $n \geq m$.

   Prove there is a unique group homomorphism $G \to \varprojlim X_n$ making all the relevant diagrams commute.
  (e) Think through (d) in the category of sets, rings, and $R$-modules.

(3) Let $(x_n)_{n \in \mathbb{N}} \in \mathfrak{C}_I(M)$ be a Cauchy sequence. Fix $t \in \mathbb{N}$.
  (a) Explain why the sequence of residues $x_n \mod I^t M$ is eventually constant as $n \to \infty$.
  (b) Use (a) to show there is a surjective group homomorphism $\mathfrak{C}_I(M) \twoheadrightarrow M/I^t M$.
  (c) Use (b) to show there is a surjective group homomorphism $\widehat{M}^I \twoheadrightarrow M/I^t M$.
  (d) Show that for any $t' \leq t$, the map for $t'$ is the composition $\widehat{M}^I \twoheadrightarrow M/I^t M \twoheadrightarrow M/I^{t'} M$.
  (e) Find a ring homomorphism $\widehat{R}^I \to \varprojlim_{n \to \infty} R/I^n$.
  (f) Find a group homomorphism $\widehat{M}^I \to \varprojlim_{n \to \infty} M/I^n M$, and show that it is $\widehat{R}^I$-linear.

(g) Prove that the maps in (e) and (f) are isomorphisms. [Hint: Given an element $\widehat{R}^I$, arbitrarily lift elements of $R/I^n$ to a sequence in $R$. Show it is Cauchy and well-defined up to $\mathfrak{C}_I^0$.]

(4) Formal Power Series. Let $S = R[x_1, ..., x_n]$ be a polynomial ring over $R$, and let $I = \langle x_1, \ldots, x_n \rangle \subset S$.

(a) Explain why an element of $S/I^n$ is uniquely represented by a polynomial of degree less than $n$ in the $x_i$.

(b) Use (a) to describe the sequences in $\Pi_{n\in\mathbb{N}} S/I^n$ that represent an element of the inverse limit $\varprojlim S/I^n$?

(c) Prove that $\varprojlim S/I^n \cong R[[x_1, ..., x_n]]$.

(5) Prove that if $R$ is a Noetherian ring, then $\widehat{R}^I$ is Noetherian.

[Hint: Map the polynomial ring $S = R[x_1, ..., x_n]$ surjectively to $R$ such that the $x_i$ map to generators $I$. Construct a surjection for the inverse limits $R[[x_1, ..., x_n]] \to \widehat{R}^I$.]

(6) Let $R = \frac{K[x_1,...,x_n]}{\langle g_1,...,g_t\rangle}$. Prove that completing at $\langle x_1, \ldots, x_n \rangle$ produces $\frac{K[[x_1,...,x_n]]}{\langle g_1,...,g_t\rangle}$.

(7) The Completion Functor.

(a) Prove that if $M \to N$ is an $R$-module homomorphism, then there is a naturally induced $\widehat{R}^I$-module homomorphism $\widehat{M}^I \to \widehat{N}^I$.

(b) Prove that completion defines a functor from $R$-modules to $\widehat{R}^I$-modules.

(c) Prove that the completion functor is right exact.

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Artin-Rees Lemma: Let $N \subset M$ be Noetherian modules over a Noetherian ring $R$ and let $I$ be an ideal of $R$. Then there is a constant positive integer $c$ such that for all $n \geq c$, $I^n M \cap N = I^{n-c}(I^c M \cap N)$. That is, eventually, each of the modules $N_{n+1} = I^{n+1}M \cap N$ is $I$ times its predecessor, $N_n = I^n M \cap N$.

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(8) Let $N \subset M$ be Noetherian modules over the Noetherian ring $R$.

(a) Assuming the Artin-Rees Lemma, show that there is a constant $c$ such that for all $n \geq c$, $I^n M \cap N \subset I^{n-c}N$ for all $n \geq c$.

(b) Show that if a sequence of elements in $N$ is an $I$-adic Cauchy sequence in $M$ (respectively, converges to 0 in $M$) then it is an $I$-adic Cauchy sequence in $N$ (respectively, converges to 0 in $N$.

(c) Prove that $\widehat{N}^I \to \widehat{M}^I$ is injective.

(d) Conclude that the **completion functor is exact on finitely generated $R$-modules.**

(9) (a) Show that there is a natural map of $R$-modules $M \to \widehat{M}^I$ whose kernel is $\bigcap I^n M$.

(b) Show that there is a natural map of $\widehat{R}^I$-modules $\widehat{R}^I \otimes_R M \to \widehat{M}^I$.

(c) Show that if $R$ and $M$ are Noetherian, then $\widehat{R}^I \otimes_R M \cong \widehat{M}^I$ as $\widehat{R}^I$-modules.

[Hint: Consider a presentation of $M$ and use right exactness of tensor and completion.]

(10) (a) Prove an $R$-module $F$ is flat if and only if whenever $N \subset M$ are finitely generated $R$-modules, the map $F \otimes_R N \to F \otimes_R M$ is injective.

(b) Deduce that $\widehat{R}^I$ **is a flat $R$-algebra when $R$ is Noetherian.**