Worksheet on Krull's Theorems.

KRULL'S INTERSECTION THEOREM. Let (R, m) be a local Noetherian ring. Then $\bigcap_{n\in\mathbb{N}} m^n = 0$.

KRULL'S PRINCIPAL IDEAL THEOREM. Let R be a Noetherian ring, and $f \in R$. Then, *every* minimal prime of $\langle f \rangle$ has height at most one.

DEFINITION. An R-module M is **simple** if the only submodules are 0 and M.

DEFINITION. An R-module M has **finite length** if it has a finite filtration $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ in which each factor M_i/M_{i-1} is simple. By the Jordan-Holder Theorem, all such filtrations have the same length n, which we call the **length** of M.

DEFINITION. An R-module M is **Artinian** if every *descending* chain of submodules eventually stabilizes.

THEOREM. The following conditions on a ring R are equivalent.

(i) R is Noetherian of Krull dimension 0.
(ii) R is a finite product of local rings of Krull dimension 0.
(iii) R has finite length as a module over itself.
(iv) R is Artinian as an module over itself (equivalently, R has DCC on ideals).

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(1) PROOF OF KRULL INTERSECTION THEOREM. Let (R, m) be a local Noetherian ring. Suppose that $\bigcap_{n\in\mathbb{N}} m^n = J$. We want to show that $J = 0$.
  (a) Explain why (two words!) it suffices to show that $mJ = J$.
  (b) Let $\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_t$ be a primary decomposition for mJ. Explain why it is enough to show that $J \subset \mathfrak{q}_i$ for all i.
  (c) Show that if $\sqrt{\mathfrak{q}_i} \neq m$, then $J \subset \mathfrak{q}_i$. [Hint: Take $x \in m \setminus \mathfrak{p}_i$ and observe $xJ \subset mJ \subset \mathfrak{q}_i$.]
  (d) Finally, show that if $\sqrt{\mathfrak{q}_i} = m$, then $J \subset \mathfrak{q}_i$. [Hint: Show there exists n such that $J \subset m^n \subset \mathfrak{q}_i$.]

(2) NAGATA'S IDEALIZATION TRICK. Let R be any commutative ring (with 1) and M any R-module.
  (a) Let $S = R \bigoplus M$ and define a multiplication by $(r \oplus m) \cdot (r' \oplus m') = rr' \oplus (rm' + r'm)$. Show that S is a commutative ring with 1.
  (b) Show that $M = 0 \oplus M \subset S$ is an ideal of S, that $M^2 = 0$ and that $S/M \cong R$.
  (c) Prove that every prime in S has the form $P \oplus M$ for some prime in R. In particular, $\operatorname{Spec} R$ and $\operatorname{Spec} S$ are homeomorphic.
  (d) Prove that if R is local, so is S.
  (e) Prove that if R and M are Noetherian, then so is S.

(3) COROLLARY OF KRULL'S INTERSECTION THEOREM. Let M be a Noetherian module over a Noetherian local ring (R, m). Prove that $\bigcap_{n\in\mathbb{N}} m^n M = 0$.
[Hint: Use Nagata's trick and apply the Krull intersection theorem to $(m \oplus M)$ in S.]

(4) Let J any ideal of R such that $\mathbb{V}(J)$ is a finite set $\{m_1, \ldots, m_t\}$ of *maximal* ideals. Prove that J has a primary decomposition as follows:

(a) Show that $\sqrt{JR_{m_1} \cap R} = \sqrt{J}R_{m_1} \cap R = (m_1 \cap m_2 \cap \cdots \cap m_t)R_{m_1} \cap R = m_1 R_{m_1} \cap R = m_1$.
(b) Set $\mathfrak{q}_i = JR_{m_i} \cap R$. Prove that $\mathfrak{q}_i$ is m_i-primary.
(c) Show that $J = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_t$. [Hint: Check $\frac{\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_t}{J} = 0$ by checking it each maximal ideal.]
(d) Explain (using a theorem you've proved!) why this decomposition is unique.
(e) Use the Chinese Remainder theorem to show also that $R/J \cong R/\mathfrak{q}_1 \times R/\mathfrak{q}_2 \times \cdots \times R/\mathfrak{q}_t$. [Hint: Check $\mathfrak{q}_i + \mathfrak{q}_j$ is the unit ideal by computing its radical.]

(5) (a) Show that M is a simple R-module iff $M \cong R/m$ for some maximal ideal m of R.
(b) Show that for any R module M, $\ell(M) = \ell(M_1) + \ell(M/M_1)$ for any submodule M_1. Here we interpret $\infty + d = \infty$.
(c) Prove that any module of finite length over R is both Noetherian and Artinian.
(d) Show that if $R = K$ is a field, then for any K-module M, finite length, Noetherian and Artinian are all equivalent.
(e) * A module M over a Noetherian ring has finite length if and only if M is finitely generated and $\text{Ass}(M)$ contains only maximal ideals.

(6) Proof of Theorem on zero dimensional rings.
(a) Prove that (i) implies (ii) by using (4) applied to the zero ideal of R.
(b) Assume (R, m) is a Noetherian local ring of dimension zero. Show that $m^n = 0$ for some m, and that $\ell(R) = \sum_{i=0}^{n} \dim_{R/m}(m^i/m^{i+1})$. [Hint: Use (5b) and the chain $R \supset m \supset m^2 \supset \cdots$.]
(c) Conclude that (ii) implies (iii).
(d) Observe that (iii) implies (iv). Hint: This is quite general; see (5c).]
(e) Show that if R is Artinian, so is every quotient ring.
(f) Show that if R is Artinian, all primes are maximal. [Hint: If P is a non-maximal prime, take non-unit non-zero element $x \in R/P$. Use the stabilizer of $\langle x \rangle \supset \langle x^2 \rangle \supset \ldots$ to show that x is a unit in R/P.]
(g) Show that if R is Artinian, then R has only finitely maximal ideals. [Hint: Look at $m_1 \supset m_1 \cap m_2 \supset \ldots$.]
(h) Complete the proof of the theorem.

(7) Consider the $\mathbb{Z}_{\langle p \rangle}$ module $\mathbb{Q}/\mathbb{Z}_{\langle p \rangle}$. Show that the classes of $\frac{1}{p^n}$ for all $n \in \mathbb{N}$ generate $\mathbb{Q}/\mathbb{Z}_{\langle p \rangle}$, and that $\mathbb{Q}/\mathbb{Z}_{\langle p \rangle}$ has DCC but not finite length. Why doesn't this contradict the Theorem?

(8) Proof of Krull Principal ideal Theorem.
(a) Show that if a counterexample exists to the Krull principal ideal theorem, then there is a counterexample in which (R, m) is a local Noetherian domain of dimension at least two and $\langle f \rangle$ is a principal ideal which has m as its unique minimal prime.
(b) With notation as in (a), prove that $\overline{R} = R/\langle f \rangle$ is zero dimensional.
(c) Let $Q \subset R$ be a non-maximal prime ideal. Show that the symbolic powers
$$Q\overline{R} \supset Q^{(2)}\overline{R} \supset Q^{(3)}\overline{R} \supset \cdots$$
eventually stabilize; say $Q^{(m)}\overline{R} = Q^{(n)}\overline{R}$ for all $m \geq n$.
(d) Prove that $Q^{(n)} = Q^{(m)} + fQ^{(n)}$ in R. [Hint: You will need to use the fact that $Q^{(n)}$ is Q-primary.]
(e) Show that $Q^{(m)} = Q^{(n)}$ for all $m \geq n$ by applying NAK to the quotient $Q^{(n)}/Q^{(m)}$.
(f) Show that if $Q \neq 0$, then $\bigcap_{m \in \mathbb{N}} Q^{(m)} \neq 0$. [Hint: Show that if $y \in Q$, then $y^n \in \bigcap_{m \in \mathbb{N}} Q^{(m)}$.]
(g) Use Krull's Intersection Theorem to show $Q = 0$. [Hint: $Q^{(m)} \subset Q^m R_Q$.]
(h) Complete the proof of Krull's Principal Ideal Theorem.

(9) *Corollary: Krull Height Theorem. Let R be a Noetherian ring. Show that every minimal prime of an ideal generated by n elements has height at most n.