Second Worksheet on the Localization

(1) Fix a ring R, multiplicative set $U \subset R$ and ideal $I \subset R$. Let $\overline{U}$ denote the image of U under the natural quotient map $R \to R/I$.
   (a) Describe the canonical isomorphism $U^{-1}R/IU^{-1}R \cong \overline{U}^{-1}R/I$ both explicitly (saying what elements correspond) and using the universal properties.
   (b) For the natural quotient map $R \to R/I$, review the induced map of Spec $R/I \to$ Spec R. Why is it injective and what is the image?
   (c) For the natural localization map $R \to U^{-1}R$, review the induced map of Spec $U^{-1}R \to$ Spec R. Why is it injective and what is the image?
   (d) For the natural localization map $R \to R_P$, review the induced map of Spec $R_P \to$ Spec R. Why is it injective and what is the image?
   (e) For the natural localization map $R \to R[\frac{1}{f}]$, explain why the induced map of spectra can be thought of as the inclusion of a basic open set $D(f) \subset$ Spec R.
   (f) Consider the natural map $R \to U^{-1}R/IU^{-1}R$, which can be viewed either as a localization followed by a quotient map or vice versa. Discuss the induced map of Spectra Spec $U^{-1}R/IU^{-1}R \to$ Spec R ? Is it injective? What is the image?
   (g) For $P \in \operatorname{Spec} R$, define the **residue field** of P to be $k(P) = R_P/PR_P$. There is a canonical map $R \to k(P)$ (why?). Describe the induced map of Spectra.

THEOREM: Let $f : Y \to X$ be the map of spectra induced by the ring homomophisms $R \xrightarrow{\phi} S$. For each $P \in X$, the fiber $f^{-1}(P) \subset Y$ can be identified with the space $\operatorname{Spec} k(P) \otimes_R S$.

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(2) FIBERS. Let $f : Y \to X$ be the map of spectra induced by each of the ring homomophisms $R \xrightarrow{\phi} S$ below. Use the Theorem to (or otherwise) describe the fibers over each point. Note that there are different kinds of points to consider.
   (a) $\mathbb{Z} \hookrightarrow \mathbb{Q}$.
   (b) $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x, y]$.
   (c) $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]$.
   (d) $\mathbb{R}[x] \hookrightarrow \mathbb{C}[x]$.

(3) Let $f : Y \to X$ be the map of spectra induced by a ring homomophisms $R \xrightarrow{\phi} S$.
   (a) Show that for any $P \in X$, the closure of P in Spec R is $\mathbb{V}(P)$.
   (b) Show that for any $P \in X$, $\operatorname{Spec} S/PS$ can be identified with $\{Q \in \operatorname{Spec} S \mid f(Q) \supset P\}$.
   (c) For any $P \in X$, let U be the image of $R \setminus P$ under the map $R \xrightarrow{\phi} S$. Prove that U is a multiplicative set in S. Now show there is a natural bijection between $\operatorname{Spec} U^{-1}S$ and the set $\{Q \in \operatorname{Spec} S \mid f(Q) \subset P\}$.
   (d) Show that for any $P \in X$, the fiber over P is homeomorphic to Spec $U^{-1}S/PU^{-1}S$ (with U as in (c)). That is, the fiber over P is $\operatorname{Spec} S \otimes_R k(P)$ where $k(P)$ is the residue field of P.

(4) For the map $\mathbb{Z} \hookrightarrow \mathbb{Z}[x, y]/\langle y^2 - x^3 - 5x + 30\rangle$, discuss the induced map of Spectra, including fibers. Are any of the fibers dramatically different from other? In what ways? What is the typical fiber like? What about the "generic fiber" (the one over the generic, or dense point of $\operatorname{Spec} \mathbb{Z}$.)

(5) For the map $\mathbb{C}[t] \hookrightarrow \mathbb{C}[x, y, t]/\langle xy - t\rangle$, discuss the induced map of Spectra, including fibers. Using **Hilbert's Nullstellensatz**, compare to a corresponding map of algebraic sets. Are there are reducible fibers? How does the fiber over the unique dense point of $\mathbb{C}[t]$ (the "generic point") compare to a "generic" or "typical" fiber?

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(6) Let U be a multiplicative set in R. For any R-module M, define $U^{-1}M$ as the set of equivalence classes of pairs $(m, u) \in M \times U$, where $(m, u) \equiv (m', u')$ when there exists $v \in U$ such that $v(u'm - um') = 0$. Prove that $M \mapsto U^{-1}M$ defines an **exact functor** from the category of R-modules to the category of $U^{-1}R$-modules. This means that if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of R-modules, then

$$0 \to U^{-1}M_1 \to U^{-1}M_2 \to U^{-1}M_3 \to 0$$

is an exact sequence of $U^{-1}R$-modules.