Worksheet on the Localization of Rings

Fix a (commutative, unital) ring R.

DEFINITION: A multiplicative set $U \subset R$ is subset closed under multiplication with $1_R \in U$.

DEFINITION: Given a multiplicative set U of R, the **localization** of R at U is a ring[1] $U^{-1}R$ satisfying the following universal property: For all ring homomorphisms $R \xrightarrow{\phi} T$ such that $\phi(U)$ consists of units in T, the map ϕ factors uniquely through $R \to U^{-1}R$. The ring $U^{-1}R$ and homomorphism $R \to U^{-1}R$ are unique up to unique isomorphism.

CONCRETELY: The localization $U^{-1}R$ is the set of equivalence classes $\frac{r}{u}$ of pairs $(r,u) \in R \times U$ under the following equivalence relation: $(r_1, u_1) \sim (r_2, u_2)$ if there exists $u \in U$ such that $u(u_2 r_1 - u_1 r_2) = 0$.

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(1) Review localization and its universal property:
  (a) What is the ring structure on $U^{-1}R$ and why is it well defined *well-defined*? Describe the maps in the factorization explicitly and understand why they are uniquely determined.
  (b) Discuss good ways to understand $U^{-1}R$ explicitly in the following cases:
    i). $R = \mathbb{Z}$, $U = \mathbb{Z} \setminus \{0\}$.
    ii). $R = \mathbb{Z}$, $U = \{2^n \,|\, n \in \mathbb{Z}_{\geq 0}\}$.
    iii). $R = \mathbb{Z}$, $U = \mathbb{Z} \setminus 2\mathbb{Z}$.
    iv). $R = \mathbb{Z}/12\mathbb{Z}$, $U = \{\overline{1}, \overline{2}, \overline{4}, \overline{8}\}$.
    v). For a fixed non-zero integer n, $R = \mathbb{Z}[x]/\langle nx - 1\rangle$, $U = \{n^k \mid k \in \mathbb{Z}_{\geq 0}\}$.
  (c) Explain how to think of $U^{-1}R$ as a quotient of the polynomial ring $R[\{X_s\}_{s \in U}]$.

(2) THE KERNEL OF LOCALIZATION
  (a). Prove that localization $R \to U^{-1}R$ is injective if and only if U contains no zero-divisors.
  (b). For a multiplicative set U, define $I := \{r \in R \,|\, ru = 0 \text{ for some } u \in U\}$. Prove that the localization map $R \to U^{-1}R$ factors through the quotient map $R \to R/I$.
  (c). Examine what this says about the example in (1) (b) (iv).

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(3) CONTRACTION AND EXPANSION. Let $R \xrightarrow{\phi} S$ be a ring homomorphism. The **contraction** of an ideal J in S, denoted $J \cap R$, is the ideal $\phi^{-1}(J) \subset R$. The **expansion** of an ideal I to S, denoted IS, is the ideal of S generated by $\phi(I)$. [These are denoted J^c and I^e, respectively, in Atiyah-MacDonald.]

  (a) Show that $(J \cap R)S \subset J$ for all ideals $J \subset S$;
  (b) Show that $I \subset (IS) \cap R$ for all ideals $I \subset R$.
  (c) Given an example of a ring map $R \to S$ and a prime ideal $P \subset R$ such that PS is not prime.
  (d) Suppose $P \in \operatorname{Spec} R$ is disjoint from a multiplicative set $U \subset R$. Prove that $PU^{-1}R$ is prime. What happens if $P \cap U \neq \emptyset$?

(4) Let $U \subset R$ be a multiplicative set.
  (a) Show that $\operatorname{Spec} U^{-1}R$ is homeomorphic to $Y = \{P \in \operatorname{Spec} R \mid P \cap U = \emptyset\}$ with the subspace topology. [Hint: the maps are given by contraction and expansion.]
  (b) For $f \in R$, show that $\operatorname{Spec} R[\frac{1}{f}]$ is homeomorphic to $D(f) \subset \operatorname{Spec} R$. [By $R[\frac{1}{f}]$, we mean the localization at the multiplicative set $\{f^n \,|\, n \in \mathbb{N}\}$.]

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[1] Or more precisely, a ring homomorphism $R \to U^{-1}R$

(c) For $P \in \operatorname{Spec} R$, prove that $U = R \setminus P$ is a multiplicative set and that $U^{-1}R$ is a **local** ring, meaning that it has a unique maximal ideal. We denote this ring R_P and call it the **localization of R at P.**

(5) (a) Show that f is nilpotent if and only if f is contained in every prime ideal of R. [Hint: Characterize nilpotency in terms of $R[\frac{1}{f}]$ and use the fact that every non-zero ring has a maximal ideal].
(b) Conclude that the nilradical of R is

$$Nil(R) = \bigcap_{P \in \operatorname{Spec} R} P = \bigcap_{P \in \operatorname{minSpec} R} P$$

(c) Conclude that R is reduced if and only if $\bigcap_{P \in \operatorname{minSpec} R} P = 0$.
(d) Conclude that for any ideal I in R, the radical $\sqrt{I} = \bigcap_{P \in \mathbb{V}(I)} P$.

(6) (a) Show that $\operatorname{Spec}(R \times S)$ is homeomorphic to the disjoint union $\operatorname{Spec} R \bigcup \operatorname{Spec} S$.
(b) Let R be the product ring $\mathbb{Z} \times \mathbb{Z}/\langle 24 \rangle \times \mathbb{Q}[x]/\langle x^2 - 1 \rangle$. Describe $\operatorname{Spec} R$.
(c) For R as in (b), let $P \in \operatorname{Spec} R$ be the prime ideal $\mathbb{Z} \times \mathbb{Z}/\langle 24 \rangle \times \langle \overline{(x-1)} \rangle$. Describe the set $\operatorname{Spec} R_P$.

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(7) (a) Discuss the following two diagrams in the category **CommRing**. Some arrows are natural quotient or localization maps; describe these with elements. Other arrows use the universal property of localizations or quotients. Explain. [Here, $\overline{U}$ denotes $\pi(U)$ where $\pi : R \to R/I$ is the natural quotient map.] To what extent are the diagrams different?

$$\begin{array}{ccc} R & \longrightarrow & U^{-1}R \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & \overline{U}^{-1}R/I \end{array} \qquad \begin{array}{ccc} R & \longrightarrow & U^{-1}R \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & U^{-1}R/IU^{-1}R \end{array}$$

(b) Prove that there is a unique isomorphism identifying residue class field of the local ring R_P at its unique maximal ideal with the fraction field of R/P. This field is called the residue field at $P \in \operatorname{Spec} R$ and denoted $k(P)$.
(c) Explain why there is a bijection between $\operatorname{Spec}(U^{-1}R/IU^{-1}R)$ and the primes in R containing I *and* disjoint from U. What does this say about $\operatorname{Spec} k(P)$?

(8) Fibers. Let $f : Y \to X$ be the map of spectra induced by the ring homomophism $R \xrightarrow{\phi} S$.
(a) Show that for any $P \in X$, the fiber over P is homeomorphic to $\operatorname{Spec}\, U^{-1}S/PU^{-1}S$ where $U = \phi(R \setminus P)$.
(b) For the map $R \to k(P)$, discuss the induced map of Spectra, including fibers.
(c) For the map $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]$, discuss the induced map of Spectra, including fibers.

(9) For the map $\mathbb{C}[t] \hookrightarrow \mathbb{C}[x, y, t]/\langle xy - t \rangle$, discuss the induced map of Spectra, including fibers. Using **Hilbert's Nullstellensatz**, compare to a corresponding map of algebraic sets. Are there are reducible fibers? How does the fiber over the unique dense point of $\mathbb{C}[t]$ (the "generic point") compare to a "generic" or "typical" fiber?

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(10) Let U be a multiplicative set in R. For any R-module M, define $U^{-1}M$ as the set of equivalence classes of pairs $(m, u) \in M \times U$, where $(m, u) \equiv (m', u')$ when there exists $v \in U$ such that $v(u'm - um') = 0$. Prove that $M \mapsto U^{-1}M$ defines an **exact functor** from the category of R-modules to the category of $U^{-1}R$-modules.