Worksheet on Properties of finitely generated K-algebras

Let K be a field.

THEOREMS FROM LAST TIME: If $K[x_1, \dots, x_d] \subset R$ is Noether Normalization of a finitely generated K-algebra R, then the Krull dimension of R is equal to d. Furthermore, if R is a **domain**, then all saturated chains of prime ideals from $\langle 0 \rangle$ to *any* maximal ideal of R have length d.

COROLLARY 1: Let R be a domain finitely generated over K. Then the dimension of R is the same as the transcendence degree of the fraction field of R over K.

COROLLARY 2: THE CATENARY PROPERTY. Let R be a finitely generated K-algebra, and take any $P, Q \in \operatorname{Spec} R$. Then every saturated chain of ideals $P = P_0 \subset P_1 \subset \cdots \subset P_h = Q$ has the same length.

THEOREM 1: Let R be a finitely generated K-algebra. Every radical ideal of R is the intersection of the maximal ideals containing it.

COROLLARY 3: Let R be a finitely generated K-algebra. The intersection of all maximal ideals containing an ideal I is equal to the radical of I.

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(1) Prove Corollary 1.

(2) Consider the ring $R = K[X, t]$ and the prime ideal $P = \langle X \rangle$. Using the results above:
  (a) Compute the dimension of R, the height of P, and the dimension of R_P.
  (b) Compute the maximal cardinality of a K-algebraically independent set in R_P.
  (c) Why does this not contradict Corollary 1?
  (d) Consider the polynomial ring $K(t)[x]$. What is the maximal cardinality of an algebraically independent set in R over $K(t)$? Over K?

(3) Find the dimensions of the following K-subalgebras of $K(x, y, z)$:
  (a) $K[x, xy, xz]$.
  (b) $K[xy, \frac{1}{xy}]$
  (c) $K[x^4, x^2y^{11}, xz^{13}, \frac{1}{xyz}]$
  (d) $K[x^2y^2, xy^2z, y^2z^2]$.

(4) PROOF OF COROLLARY 2. Let R be a finitely generated K-algebra.
  (a) Fix prime ideals $P \subset Q$ of R. Given a saturated chain of primes from P to Q of length c, prove there is a saturated chain of primes in the domain R/P of length c from $\langle 0 \rangle$ to Q/P.
  (b) Prove Corollary 2 follows from the THEOREM above proved last time.
  (c) Assume R is a *domain*. For $P \in \operatorname{Spec} R$. Prove that height $P = \dim R - \dim(R/P)$.
  (d) Find a counterexample to (c) when R is not a domain. [Hint: You played with one last time.]

(5) Consider the following examples. Why do they not contradict Theorem 1?
  (a) Let $R = K[x, y]/\langle x^3 \rangle$. Show that the ideal $\langle \overline{x}^2 \rangle$ is **not** the intersection of all the maximal ideals containing it.

(b) Let $R = K[x,y]_m$ where m is an arbitrary maximal ideal of $K[x,y]$. Show that most radical ideals are not the intersection of all maximal ideals containing them. Indeed, only one ideal has this property: which one?

(6) Polynomials as Functions. **Lemma:** *Assume K is infinite. For any non-zero polynomial in $K[x_1, \ldots, x_d]$, the function $K^d \to K$ sending $p \mapsto f(p)$ is a non-zero function of K^d.*
   (a) Prove the lemma in the case $d = 1$.
   (b) Use induction to prove the lemma in general. [Hint: Write $f \in K[x_1, \ldots, x_d]$ as a polynomial in one variable with coefficients polynomials in the others.]
   (c) For K finite, find a non-zero polynomial in $K[x]$ which determines the zero function on K.

(7) Proof of Theorem 1. Let R be a finitely generated K-algebra.
   (a) Show that to prove Theorem 1, it suffices to prove that an arbitrary *prime ideal* P is an intersection of maximal ideals.
   (b) Observe that $P \subset \bigcap_{P \subset m} m$.
   (c) Reduce the proof of Theorem 1 to the following statement: *If R is a finitely generated domain, and u is in every maximal ideal of R, then $u = 0$.* [Hint: If $u \notin P$ but u is in every maximal ideal containing P, look at $u + P$ in R/P.]
   (d) Using Noether Normalization, further reduce the proof of Theorem 1 to the statement in (c) for $R = K[x_1, \ldots, x_d]$. [Hint: You've shown before that if $A \subset B$ is an integral extension of domains, then every non-zero element of B has a non-zero multiple in A.]
   (e) Let $\overline{K}$ be the algebraic closure of K and consider the ring extension $K[x_1, \ldots, x_d] \subset \overline{K}[x_1, \ldots, x_d]$. Prove that if u is in every maximal ideal of $K[x_1, \ldots, x_d]$, then u is in every maximal ideal of $\overline{K}[x_1, \ldots, x_d]$. Use this to further reduce to the case $K = \overline{K}$.
   (f) Use (6) and the Nullstellensatz to complete the proof of Theorem 1.

(8) Let R be a Noetherian ring.
   (a) Show that an arbitrary intersection of radical ideals is radical.
   (b) Prove the Corollary to Theorem 1. [Hint: Use the fact that $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$.]

(9) Show that a point $P \in \operatorname{Spec} R$ is closed if and only if P is maximal.

(10) Let R be a finitely generated K-algebra. Consider $\operatorname{maxSpec} R$ as a subspace of $\operatorname{Spec} R$ with the subspace topology.
   (a) For any ideal $I \subset R$, show that $\mathbb{V}(I) \cap \operatorname{maxSpec} R$ is *dense* in $\mathbb{V}(I)$. [Hint: Reduce to I radical.]
   (b) Show that there is a bijection between closed sets of $\operatorname{Spec} R$ and closed sets of $\operatorname{maxSpec} R$ given by intersection with the subspace $\operatorname{maxSpec} R$.
   (c) Give an example of a ring R for which intersection with the subspace $\operatorname{maxSpec} R$ does *not* define a bijection between closed sets of $\operatorname{Spec} R$ and $\operatorname{maxSpec} R$. Indeed, often $\operatorname{maxSpec} R$ contains almost no information at all about $\operatorname{Spec} R$. Explain.

(11) Assume K is algebraically closed, and let $V \subset K^n$ be an algebraic set, say $V = \mathbb{V}(I)$ for some radical ideal $I \subset K[x_1, \ldots, x_n]$. Recall that V has the structure of a topological space in which the closed set are sub-algebraic sets. Let $R = K[V]$ be its coordinate ring.
   (a) Describe the image of the map $V \to \operatorname{Spec} R$ sending $p \mapsto m_p := \{f \in R \mid f(p) = 0\}$. Is this map injective? Is it continuous? Explain why it induces a bijection between open sets.
   (b) Let $V = \mathbb{V}(xy) \subset K^2$. For the map $V \to \operatorname{Spec} R$ in (b), describe the points in $\operatorname{Spec} R$ not in the image. Are any of these closed points?
   (c) For $V = K^2$, describe the points of $\operatorname{Spec} K[x,y]$ not in the image of the map $V \to \operatorname{Spec} R$ as in (b). Are any of these closed points?