Worksheet on Affine Noetherian Schemes

Let R be a commutative ring with 1.

DEFINITION: A topological space is **irreducible** if it can *not* be written as the union of two proper closed subspaces. An **irreducible component** of a topological space is a closed irreducible subset which is *maximal* with respect to inclusion.

THEOREM 1. The irreducible components of $\operatorname{Spec} R$ are the closed sets $\mathbb{V}(P)$ where P is a *minimal prime* of R, and $\operatorname{Spec} R$ is the union of these irreducible components. In particular, if R is a domain, $\operatorname{Spec} R$ is irreducible.

THEOREM 2. Every ideal in a Noetherian ring has finitely many minimal primes. In particular, a Noetherian ring has finitely many minimal primes.

COROLLARY 2. If R is Noetherian, then $\operatorname{Spec} R$ has finitely many irreducible components.

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(1) Review the following facts about the Zariski topology on $\operatorname{Spec} R$:
   (a) The closed sets are $\mathbb{V}(\{f_\lambda\}_{\lambda\in\Lambda}) = \mathbb{V}(\langle\{f_\lambda\}_{\lambda\in\Lambda}\rangle) = \mathbb{V}(\sqrt{\langle\{f_\lambda\}_{\lambda\in\Lambda}\rangle})$.
   (b) For any subsets $I \subset J$ of R, $\mathbb{V}(J) \subset \mathbb{V}(I)$.
   (c) $\bigcap_{\lambda\in\Lambda} \mathbb{V}(I_\lambda) = \mathbb{V}(\bigcup_{\lambda\in\Lambda} I_\lambda)$.
   (d) $\bigcup_{i=1}^t \mathbb{V}(I_i) = \mathbb{V}(I_1 \cap \cdots \cap I_t) = \mathbb{V}(I_1 \cdot \cdots \cdot I_t)$.
   (e) If $N \subset R$ denotes the nilradical of R, then the closed set $\mathbb{V}(N)$ is equal to $\operatorname{Spec} R$.
   (f) For any ideal I, contraction for the quotient map $R \to R/I$ induces a homeomorphism $\operatorname{Spec} R/I \cong \mathbb{V}(I)$, where $\mathbb{V}(I) \subset \operatorname{Spec} R$ has the subspace topology.

(2) Using the theorems above, which of the following are irreducible topological spaces? Count the components of those that are not.
   (a) $\operatorname{Spec} \mathbb{Z}$
   (b) $\operatorname{Spec} \mathbb{Z}/\langle 24\rangle$
   (c) $\operatorname{Spec} \mathbb{Z}/\langle 25\rangle$
   (d) $\operatorname{Spec} K[x,y,z,w]/\langle xy^5 + z^3y^2 + wx\rangle$. [Hint: Use Eisenstein with the prime $\langle z,w\rangle$.]
   (e) $\operatorname{Spec} K[x,y,z,w]/\langle xy, xz, xw\rangle$.
   (f) $\operatorname{Spec}(L_1 \times L_2)$ where L_1, L_2 are fields.
   (g) $\operatorname{Spec} R$ where R is the product, over all positive prime integers p, of the rings $\mathbb{Z}/p\mathbb{Z}$.

(3) PROOF OF THEOREM 1. Let R be arbitrary.
   (a) Prove that if $\operatorname{Spec} R = \mathbb{V}(x)$, then x is nilpotent.
   (b) Suppose that R is reduced. Show that if $\operatorname{Spec} R$ is irreducible, then R is a domain. [Hint: Recall that $\mathbb{V}(xy) = \mathbb{V}(x) \cup \mathbb{V}(y)$.]
   (c) Assume R is a domain. Prove that $\operatorname{Spec} R$ is irreducible. [Hint: What closed sets contain $\langle 0\rangle$?]
   (d) Show that every irreducible closed set of $\operatorname{Spec} R$ has the form $\mathbb{V}(P)$ where $P \in \operatorname{Spec} R$.
   (e) Prove that the irreducible components of $\operatorname{Spec} R$ are of the form $\mathbb{V}(P)$ where P is a *minimal* prime of R.
   (f) Prove that $\operatorname{Spec} R = \bigcup_{P\in \min\operatorname{Spec} R} \mathbb{V}(P)$.

(4) Proof of Theorem 2. We'll use *Noetherian Induction.*
  (a) Explain why Theorem 2 is equivalent to the (*a priori* weaker) statement that every Noetherian ring has finitely many minimal primes.
  (b) To prove Theorem 2, fix a Noetherian ring R. Consider the set of all ideals in R that have infinitely many minimal primes. Show that this set has a maximal element (with respect to inclusion) if it is non-empty.
  (c) To prove Theorem 2, show that it suffices to prove it for R with the property that every proper quotient has finitely many minimal primes. [Hint: Use (b).]
  (d) Explain why, if R in (c) is a domain, the proof of Theorem 2 is complete.
  (e) With R as in (c), suppose $x, y \in R$ are non-zero elements such $xy = 0$. Show that every minimal prime of R contains either x or y (or both).
  (f) Again, with R as in (c), show that $\langle x \rangle$ and $\langle y \rangle$ have only finitely many minimal primes. [Hint: Use the Noetherian induction hypothesis (c).]
  (g) Prove Theorem 2 and its corollary.

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(5) Lemma. Fix a vector space over an infinite field. Let $\{W_1, \dots, W_n\}$ be a finite collection of vector subspaces. Prove that if V is a subspace contained in $\bigcup_{i=1}^n W_i$, then $V \subset W_i$ for some i. [Hint: Say $x \in V$, and wlog, $x \in W_1$. Pick $y \in V \setminus W_1$. Consider the elements $x + ay$ where $a \in K \setminus \{0\}$. Induce.]

(6) In the ring $\mathbb{C}[x, y, z]$, let $U = \mathbb{C}[x, y, z] \setminus (\langle x \rangle \bigcup \langle y, z \rangle)$. Let $R = U^{-1}\mathbb{C}[x, y, z]$. Describe Spec R: what are the maximal and minimal primes? How many components? What is the dimension? What are the heights of the different maximal ideals? How does this look as a poset? as a subset of Spec $\mathbb{C}[x, y, z]$? What is its closure in Spec $\mathbb{C}[x, y, z]$?
  [Hint: Use the Lemma in (5)! Don't forget that ideals in a K-algebra are also K-subspaces.]

(7) Prime Avoidance Lemma. Let R be any ring, and let $I_1, \dots, I_t$ be ideals of R. Suppose that an ideal $J \subset I_1 \cup I_2 \cup \cdots \cup I_t$.
  (a) If R is an algebra over an infinite field, prove that $J \subset I_k$ for some index k. [Hint: Use (5).]
  (b) * More generally, assume most two of the ideals I_k are not prime. Prove that $J \subset I_k$ for some index k.

(8) * Consider a doubly indexed set of variables $\{x_{ij} \mid i \leq j,\ i, j \in \mathbb{N}\}$. Let S be the polynomial ring they generate over $\mathbb{C}$, so $S = \mathbb{C}[x_{11}, x_{12}, x_{22}, x_{13}, x_{23}, x_{33}, \dots]$. For each fixed j, let P_j be the prime ideal generated by $\{x_{1j}, x_{2j}, \dots, x_{jj}\}$. Let $U = S \setminus \bigcup_{j=1}^{\infty} P_j$.
  (a) Show that $U = S \setminus \bigcup_{n=1}^{\infty} P_n$ is a multiplicative set. Let $R = U^{-1}S$.
  (b) ** Show that if an ideal $I \subset S$ is contained in $\bigcup_{n=1}^{\infty} P_n$, then $I \subset P_n$ for some n. [Hint: for $f \in I$, consider the (non empty, finite) set $Q(f) := \{i \in \mathbb{N} \mid f \in P_i R\}$. Show we're done unless $\forall f \in I, \exists g \in I$ such that $Q(f) \cap Q(g) = \emptyset$. Now look at $f + x_m^d g$ (which is in I) for well-chosen $m \in Q(g)$ and $d \gg 0$.]
  (c) Prove that the maximal ideals of $R = U^{-1}S$ are precisely the $P_j R$.
  (d) Show that R has chains of primes of arbitrarily long length.
  (e) Prove that the localization of R at any maximal ideal is Noetherian.
  (f) Prove that any non-zero $f \in R$ is contained in at most finitely maximal ideals of R.
  (g) * Prove that a ring is Noetherian if its localization at any maximal ideal is Noetherian and any non-zero f is contained in only finitely many ideals.
  (h) Prove that $U^{-1}R$ is Noetherian but has infinite Krull dimension.