Worksheet on Nakayama's Lemma

Let (R, m) be a local[1] ring with maximal ideal m.

NAKAYAMA'S LEMMA. Let M be a finitely generated module over (R, m). If $mM = M$, then $M = 0$.

NAKAYAMA'S LEMMA, VERSION 2. Let M be finitely generated over (R, m). A subset $\{x_1, \ldots, x_d\} \subset M$ is a *generating set* for M if and only if their images span M/mM as a vector space over the field R/m.

THEOREM. Let M be a finitely presented module over an arbitrary ring S. Then the following conditions are equivalent:

(1) M is projective.
(2) M is flat.
(3) M is locally free (meaning that M_P is free for all $P \in \operatorname{Spec} R$, or equivalently, all $P \in \operatorname{maxSpec} R$).

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(1) Let M be a module over (R, m).
   (a) Verify that M/mM really is a vector space over R/m.
   (b) Show that $M/mM \cong R/m \otimes_R M$.
   (c) Compute the R/m-dimension of M/mM in the following cases:
      (i) $R = K[x, y]_{\langle x,y\rangle}$ and $M = R^{\oplus 5}$.
      (ii) $R = K[x_1, x_2, \ldots, x_n]_m$ where $m = \langle x_1, \ldots, x_n\rangle$, and $M = m$.
      (iii) $R = \frac{K[[x,y,z]]}{\langle x^4+y^4+z^4\rangle}$ and $M = m$.

(2) Let $R = K[[x]]$ and let M be the R-module $K[[x]][x^{-1}]$. Show that $mM = M$ for the ideal $m = \langle x\rangle$. Why doesn't this contradict Nakayama's Lemma?

(3) PROOF OF NAKAYAMA'S LEMMA. Fix M finitely generated over (R, m).
   (a) Prove Nakayama's Lemma in the case M is generated by one element.
   [Hint: If x generates M, show there exists $r \in m$, such that $rx = x$; now observe that $r - 1$ is a unit.]
   (b) Show that if M is generated by $x_1, \ldots, x_n$ and $mM = M$, then $M' = M/Rx_n$ is generated by $n - 1$ elements and $M' = mM'$.
   (c) Prove Nakayama's Lemma by induction on the number of generators of M.
   [Hint: For the inductive step, use (b) and then (a).]

(4) THE SECOND VERSION OF NAKAYAMA'S LEMMA. Let M be finitely generated over (R, m) and fix $\{x_1, \ldots, x_d\} \subset M$. Assume that $\{\overline{x_1}, \ldots, \overline{x_d}\} \subset M/mM$ spans M/mM over R/m. Use the first version of Nakayama's Lemma to show that $\{x_1, \ldots, x_d\}$ generate M.
[Hint: Let $N \subset M$ be generated by $\{x_1, \ldots, x_d\}$. Show $M = N + mM$, and that this implies $M/N = m(M/N)$.]

(5) Show that every minimal set of generators for a finitely generated module over a local ring has the same cardinality. [Minimal generating set means no proper subset also generates.]

(6) Let (R, m) be a local domain that is not a field. Prove that the fraction field of R is not a finitely generated R-module. [Hint: What would Nakayama say about a minimal generating set?]

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[1]We do not assume R is Noetherian. Note: Mel's (somewhat non-standard) terminology for a ring with unique maximal ideal is *quasilocal*. He reserves the word "local" for *Noetherian rings* with unique maximal ideal.

(7) Suppose $IM = M$ for some finitely generated R-module M and proper ideal I in a local ring. Prove that $M = 0$.

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(8) Let $M \xrightarrow{g} N$ be a morphism of finitely generated modules over a local ring (R, m).
   (a) Prove g is surjective if and only if the induced map of R/m vector spaces $M/mM \xrightarrow{g} N/mN$ is surjective. [Hint: Consider the right exact sequence $M \xrightarrow{g} N \to \operatorname{coker} g \to 0$. Use (1b).]
   (b) Show that both implications of the corresponding statement about injectivity are false. [Hint: Let $R = \mathbb{Z}_{\langle p \rangle}$. Consider $R \xrightarrow{\times p} R$ and $R \twoheadrightarrow R/\langle p \rangle$.]

(9) Projective vs free modules. Let M a finitely generated module over a local ring (R, m).
   (a) Suppose that $x_1, \dots, x_n \in M$ is a minimal generating set for M. Explain why there is a surjection $R^{\oplus n} \xrightarrow{\pi} M$ sending $e_i \mapsto x_i$.
   (b) Assume is M is projective. Show there is an R-linear map $M \xrightarrow{\sigma} R^{\oplus n}$ such that the composition $M \xrightarrow{\sigma} R^{\oplus n} \xrightarrow{\pi} M$ is the identity map.
   (c) Prove σ is an isomorphism. [Hint: Apply the functor $R/m \otimes_R -$ and use Nakayama's Lemma.]
   (d) Prove that a finitely generated module over a local ring is projective if and only it if is free.
   (e) Prove that a finitely presented module M over an arbitrary ring is projective if and only if M is locally free. [Hint: You proved last time that projectivity is local for finitely presented modules.]

(10) Let $S = \mathbb{Z}[\sqrt{-5}]$. Let $m = \langle 2, 1 + \sqrt{-5} \rangle$.
   (a) Observe that $S = \mathbb{Z}[x]/\langle x^2 + 5 \rangle$, and that m is a maximal ideal.
   (b) Explain why the ideal $\langle 17 \rangle \subset S$ is a projective S-module. [Hint: $\langle 17 \rangle$ is principal in a domain!]
   (c) Show that m is not principal. [Hint: This is easy if you use the norm[2] $N(a + b\sqrt{-5}) = a^2 + 5b^2$.]
   (d) Prove mS_m is principal in the local ring S_m. [Hint: $(1 + \sqrt{-5})(1 - \sqrt{-5}) = 2 \cdot 3$.]
   (e) Prove that if $P \in D(2) \bigcup D(1 + \sqrt{-5})$, then $m_P = S_P$. [Hint: Do one at a time.]
   (f) Show that m is the only point of $\operatorname{Spec} S$ not in $D(2) \bigcup D(1 + \sqrt{-5})$.
   (g) Prove the S-module m is projective but not free. [Hint: Use 9(e). If free, what is the rank?]

(11) Flat versus free modules. Let M be a module over an arbitrary ring S.
   (a) Show M is flat if and only if M_P is flat over S_P for all $P \in \operatorname{Spec} S$ (or all $P \in \operatorname{maxSpec} S$). [Hint: Recall that exactness is local and note $U^{-1}R \otimes_R (M_1 \otimes_R M_2) \cong (U^{-1}M_1) \otimes_{U^{-1}R} (U^{-1}M_2)$.]
   (b) Show that if M is flat over (R, m), then the map $m \otimes_R M \to M$ sending $r \otimes x \mapsto rx$ is injective.
   (c) * Show that a finitely presented flat module over a local ring (R, m) is free. [Hint: If M is minimally generated by n elements, we have an exact sequence $0 \to N \to R^n \to M \to 0$. Build a diagram from this by tensoring with each term in the seqiuence $0 \to m \to R \to R/m \to 0$.]
   (d) Prove the Theorem on the equivalence of flat, projective and locally free for f. p. modules.

(12) Show $\mathbb{Q}$ is a flat $\mathbb{Z}$-module which is not locally free. Why doesn't this contradict the Theorem?

(13) Nakayama's Lemma for arbitrary rings. Let S be an arbitrary ring, and let J be an ideal contained in every maximal ideal of S (that is, contained in the *Jacobsen radical* $\bigcap_{m \in \operatorname{maxSpec} S} m$.)
   (a) Show that if $JM = M$ for some finitely generated S-module M, then $M = 0$.
   (b) Let $S = U^{-1}K[x, y, x, z]$ where $U = K[w, x, y, z] \setminus (P_1 \bigcup P_2)$ where $P_1 = \langle x, y \rangle, P_2 = \langle z, w \rangle$. Show that $J = P_1 S \cap P_2 S$ is contained in every maximal ideal of S.
   (c) If R is a finitely generated algebra over a field K, what ideals J have the property that $J \subset m$ for all $m \in \operatorname{maxSpec} R$? What is the Jacobsen radical of R?

[2] See e.g. Dummit and Foote, §8.1, especially Example (2) on page 273 in my 2004 edition.