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Worksheet on Noether Normalization

DEFINITION: A set of elements $\{a_1, \dots, a_t, \dots\}$ in an R-algebra A is **algebraically independent over** R if every *algebraic relation* on the a_i over R is trivial—that is, given any polynomial $f(x_1, \dots, x_t) \in R[x_1, \dots, x_t, \dots]$ such that $f(a_1, \dots, a_t) = 0$, it follows that $f = 0$ as a polynomial in $R[x_1, \dots, x_t, \dots]$.

NOETHER NORMALIZATION THEOREM: Let R be a finitely generated K-algebra. Then R is a module finite extension of a polynomial subring over K in finitely many variables. That is, there exist algebraically independent elements $x_1, \dots, x_d \in R$ such that $K[x_1, \dots, x_d] \subset R$ is a module extension.

GENERAL FORM OF NOETHER NORMALIZATION THEOREM: Let R be an arbitrary domain, and let A be a finitely generated R-algebra. Then there exist non-zero $c \in R$ such that $A[\frac{1}{c}]$ is a module finite extension of a polynomial ring over $R[\frac{1}{c}]$.

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(1) EQUIVALENT FORMULATIONS OF ALGEBRAIC INDEPENDENCE. Let $\mathcal{S} = \{y_1, \dots, y_t, \dots\}$ be a subset of an R-algebra A. Show the following are equivalent:
  (a) The set $\mathcal{S}$ is algebraically independent.
  (b) The R-algebra homomorphism
  $$R[x_1, \dots, x_t, \dots] \longrightarrow A \quad x_i \mapsto y_i$$
  is injective.
  (c) The R-subalgebra of A generated by $\mathcal{S}$ is a polynomial ring over R (with "indeterminates" the elements of $\mathcal{S}$).
  (d) The collection of all monomials $y_1^{n_1} \dots y_t^{n_t}$ (where $y_i \in \mathcal{S}$ and $n_k \in \mathbb{Z}_{\geq 0}$) is **linearly independent** over R.

(2) Find two different polynomial sub-algebras over which each finitely generated $\mathbb{Q}$-algebra is module finite.
  (a) $\mathbb{Q}[x, y, z]$.
  (b) $\mathbb{Q}[x, y, z]/\langle z^2 - xy\rangle$.
  (c) $\mathbb{Q}[x, y]/\langle xy\rangle$. [Hint: Try a simple linear change of coordinates].

(3) Let $A = R[x_0, x_1 \dots, x_n]$ be a polynomial ring over R. For any $g_1, \dots, g_n \in R[x_0]$, consider the map
$$R[x_0, \dots, x_n] \xrightarrow{\phi} R[x_0, \dots, x_n] \qquad x_i \mapsto \begin{cases} x_0 & i = 0 \\ x_i + g_i(x_0) & i > 0 \end{cases}.$$
  (a) Show that ϕ is an R-algebra automorphism.
  (b) When each g_i is some power of x_0, show that $\phi(x_0 \dots x_n)$ is **monic** as a polynomial in x_0.
  (c) When each $g_i = x_0^{D^i}$ for some $D > 0$, show that $\phi(r x_0^{a_0} x_1^{a_1} \dots x_n^{a_n})$ has leading term
  $$r x_0^{a_0 + a_1 D + a_2 D^2 + \cdots + a_n D^n}$$
  as a polynomial in x_0.
  (d) Fix $f \in R[x_0, \dots, x_n]$, and choose D larger than any exponent a_i on any x_i appearing in f. With g_i as in (c), show that $\phi(f)$ has leading term of the form $r x_0^M$ as a polynomial in x_0 for some $M \in \mathbb{N}$ and non-zero $r \in R$. [Hint: Use the uniqueness of the Base D expansion of a natural number to show there is no troublesome cancellation.]

(4) PROOF OF NOETHER NORMALIZATION: Let R be an arbitrary domain, and let A be a finitely generated R-algebra. We will prove the general Noether Normalization statement by induction on the number of generators for the R-algebra A.
 (a) Check the case where the empty set generates A over R.
 (b) Assume we have proven the statement for R-algebras generated by n elements. Let A be generated over R by $\theta_0, \ldots, \theta_n$. Explain why the statement holds for A if the θ_i are algebraically independent.
 (c) If $\{\theta_0, \ldots, \theta_n\}$ satisfy some polynomial in $R[x_0, \ldots, x_n]$, use (3) to show how to replace the θ_i by some other set of generators for A over R which satisfy a *monic* polynomial in θ_0 with coefficients in $R[\frac{1}{c}][\theta_1, \ldots, \theta_n]$ for some non-zero c.
 (d) Use induction to complete the proof of the general statement of Noether Normalization.
 (e) Deduce Noether Normalization over a field.

(5) Let L be a field extension of K, and assume L is finitely generated as an K-algebra. Use Noether Normalization to prove that L is a finite algebraic extension of K. [Hint: Recall that you've found a lower bound on the dimension of a polynomial ring over a field.]

(6) THE NULLSTELLENSATZ, REVISITED. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over an algebraically closed field K. Let m be a maximal ideal of R. Prove that m can be generated by elements of the form $x_i - \lambda_i$ for $\lambda_i \in K$. [Hint: Consider the extension of fields $K \hookrightarrow R \twoheadrightarrow R/m$; Use (5)]

This completes the proof of the Nullstellensatz in general! Recall that you had proved it in the special case where K is uncountable, and you had reduced the general case to checking that maximal ideals have this special form.

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(7) Let R and S be finitely generated K-algebras, and let $R \longrightarrow S$ be a K-algebra homomorphism.
 (a) Prove that every maximal ideal of $\operatorname{Spec} S$ contracts to a maximal ideal of R. That is, the map of Spectra restricts to a map of maxSpec. [Hint: Consider $K \hookrightarrow R/(m \cap R) \hookrightarrow S/m$]
 (b) Give an example of a ring map for which some maximal ideal contracts to a non-maximal ideal.

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(8) SPECIAL FORM OF NOETHER NORMALIZATION THEOREM: Let K be an *infinite* field, and say R is a K-algebra finitely generated by $r_1, \ldots, r_n$. Then there exist a set of algebraically independent elements $y_1, y_2, \ldots, y_d$ such that
 (a) each y_i is a K-linear combination of the generators $r_1, \ldots, r_n$.
 (b) $K[y_1, y_2, \ldots, y_d] \subset R$ is a module finite extension.

Prove this as follows:
 (a) With notation as in (3), consider the K-algebra automorphism with each $g_i = \lambda_i x_0$, for some $\lambda_i \in K$. Consider $\phi(\lambda x_0^{a_0} x_1^{a_1} \ldots x_n^{a_n})$ as a polynomial in x_0 with coefficients in $K[x_1, \ldots, x_n]$. Compute the coefficient of the leading term (assuming it is non-zero).
 (b) Fix arbitrary $f \in K[x_0, \ldots, x_n]$. Consider $\phi(f)$ as a polynomial x_0 with coefficients in $K[x_1, \ldots, x_n]$. Show that the the leading term is $f_n(1, \lambda_1, \ldots, \lambda_n)$ where f_n is the largest degree piece of f. That is, writing f uniquely as a sum $\sum f_i$ where f_i is homogeneous of degree i, the element f_n is the largest degree non-zero polynomial which appears.
 (c) Explain why, if K is infinite, we can find λ_i so that $f_n(1, \lambda_1, \ldots, \lambda_n)$. Indeed, most choices of λ_i will work: why?
 (d) Prove the Special form above. Explain the significance when R is a **graded** K-algebra.