Worksheet on Noetherian rings and modules

Definition: A ring R is **Noetherian** if every ascending chain of ideals of R,

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

eventually stabilizes: there is some N for which $I_n = I_{n+1}$ for all $n \geq N$.

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(1) Show that each of the following is equivalent to R being Noetherian:
  a) Every nonempty family of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ has a maximal element[1].
  b) Every ascending chain of *finitely generated* ideals eventually stabilizes.
  c) Given any generating set S for an ideal I, the ideal I is generated by a finite subset of S.
  d) Every ideal of R is finitely generated.

(2) a) Show that if R is Noetherian and $I \subseteq R$ is an ideal, then R/I is Noetherian.
  b) Show that if $R/(f)$ is Noetherian for all $f \in R \smallsetminus \{0\}$, then R is Noetherian.

(3) Show that fields and principal ideal domains are Noetherian.

(4) Which of the following rings are Noetherian?
  (a) $\mathbb{Q}$
  (b) $\mathbb{Z}$
  (c) $\mathbb{R}[x]$
  (d) $\mathcal{C}(\mathbb{R},\mathbb{R})$, the ring of continuous real-valued functions in one variable.
  (e) $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, the ring of smooth real-valued functions in one variable.
  (f) $\mathbb{C}\{z\}$, the subring of $\mathbb{C}[\![z]\!]$ consisting of functions holomorphic in a neighborhood of $0 \in \mathbb{C}$.
  (g) The polynomial ring $\mathbb{C}[x_1, x_2, x_3, \dots]$ in countably many variables.

(5*) Show that R is Noetherian if and only if every prime ideal of R is finitely generated.

(6*) If the set of prime ideals in R satisfies ACC, must R be Noetherian?

Definition: An R-module M is **Noetherian** if every ascending chain of submodules of M,

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

eventually stabilizes: there is some N for which $M_n = M_{n+1}$ for all $n \geq N$.

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(7) Show that R is a Noetherian ring if and only if R is a Noetherian R-module.

(8) Convince yourself that essentially the same argument as in (1) shows that each of the following is equivalent to M being Noetherian:
  a) Every nonempty family of submodules $\{M_\lambda\}_{\lambda\in\Lambda}$ has a maximal element.
  b) Every ascending chain of *finitely generated* submodules eventually stabilizes.
  c) Given any generating set S for a submodule N, the submodule N is generated by a finite subset of S.
  d) Every submodule of M (including M itself!) is finitely generated.

[1] This means that there is some $\gamma \in \Lambda$ such that $I_\gamma \subseteq I_\lambda$ implies $I_\gamma = I_\lambda$ for any $\lambda \in \Lambda$.

(9) Let $N \subseteq M$ be R-modules.
a) Show that if M is Noetherian, then N and M/N are both Noetherian.
b) Use the Lemma below to show that if N and M/N are Noetherian, then M is Noetherian.

LEMMA: Let M be a module, and $M' \subseteq M''$ and N all be submodules of M. Then $M' = M''$ if and only if $M' \cap N = M'' \cap N$ and $M'/(M' \cap N) = M''/(M'' \cap N)$.

(10) Let R be a Noetherian ring, and M an R-module. Show that M is Noetherian if and only if it is finitely generated. Conclude that in a Noetherian ring, every submodule of a finitely generated module is finitely generated.

(11*) Find examples of each of the following: a ring R and R-module M such that
a) R and M are both Noetherian;
b) R is Noetherian and M is not;
c) R is not Noetherian and M is;
d) neither R nor M is Noetherian.

THEOREM (THE HILBERT BASIS THEOREM): If R is Noetherian, then $R[x]$ is Noetherian.

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(12) This problem outlines a proof of the Hilbert Basis Theorem.
a) For a polynomial $f(x) = r_n x^n + r_{n-1}x^{n-1} + \cdots + r_0$, with $r_i \in R$ and $r_n \neq 0$, we set $\mathrm{LT}(f) = r_n \in R$. Show that if $I \subset R[x]$ is an ideal, then $\mathrm{LT}(I) := \{\mathrm{LT}(f) \mid f \in I\}$ is a (possibly improper) finitely generated ideal of R.
b) Pick $f_1, \ldots, f_t \in I$ such that $\mathrm{LT}(I) = (\mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_t))$, and set $N = \max_i\{\deg(f_i)\}$. Show that every element $f \in I$ can be written as $f = \sum_i r_i f_i + g$ with $g \in I$ of degree less than N.
c) Show that the set of $g \in I$ of degree less than N is a finitely generated R-module.
d) Finish the proof.

(13) Prove the corollary: if R is Noetherian, then every finitely generated R-algebra is Noetherian. In particular, finitely generated algebras over fields are Noetherian.