Worksheet on Normal domains

Let R be a domain.

DEFINITION. The **normalization** of R is the integral closure of R in its fraction field. On this worksheet, we let $\widetilde{R}$ denote the normalization of R. A domain R is **normal** if $\widetilde{R} = R$.

THEOREM 1. A one dimensional Noetherian local ring is a DVR.

THEOREM 2. Let R be Noetherian and normal, $f \in R \setminus \{0\}$. Every associated prime of $\langle f \rangle$ has height one.

DEFINITION. A **divisor** on $\operatorname{Spec} R$ is a formal $\mathbb{Z}$-linear combination of height one primes of R. The set of all divisors on $\operatorname{Spec} R$ form a free abelian group Div (R).

DEFINITION. Let R be a Noetherian normal domain. For non-zero $f \in R$ define the **divisor** of f to be the formal sum $\sum_{i=1}^{t} a_i[P_i]$ where $P_1, \ldots, P_t$ are the minimal primes of $\langle f \rangle$ and $a_i = \nu_{P_i}(f)$ where ν_{P_i} is the valuation of the DVR R_{P_i}.

DEFINITION. The **divisor class group**[1] of $\operatorname{Spec} R$, denoted $Cl(R)$ is the group Div (R) modulo the subgroup generated by divisors of non-zero elements $f \in R$.

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(1) NORMALIZATION COMMUTES WITH LOCALIZATION. Let R be a domain and $U \subset R$ any multiplicative subset.
  (a) Be sure you understand why R and $U^{-1}R$ have the same fraction field. Call it K.
  (b) Prove that if $\frac{a}{b} \in K$ is integral over R, then $\frac{a}{b}$ is integral over $U^{-1}R$.
  (c) Conclude that $\widetilde{R} \subset \widetilde{U^{-1}R}$ and also $U^{-1}\widetilde{R} \subset \widetilde{U^{-1}R}$ as subsets of K.
  (d) Show that if $\frac{a}{b} \in K$ satisfies an equation of integral dependence
$$X^n + \frac{r_1}{u_1}X^{n-1} + \cdots + \frac{r_{n-1}}{u_{n-1}}X + \frac{r_n}{u_n} \in U^{-1}R[X]$$
  over $U^{-1}R$, then $\frac{ua}{b} \in K$ is integral over R where $u = \Pi_{i=1}^{n} u_i$.
  (e) Prove that $U^{-1}\widetilde{R} = \widetilde{U^{-1}R}$. That is, *normalization commutes with localization.*

(2) NORMALITY IS A LOCAL PROPERTY. Prove the following are equivalent for a domain R.
  (i) R is normal.
  (ii) $U^{-1}R$ is normal for all multiplicative sets $U \subset R$.
  (iii) R_P is normal for all $P \in \operatorname{Spec} R$.
  (iv) R_m is normal for all $m \in \operatorname{maxSpec} R$.
  [Hint: For (iv) implies (i), check the triviality of the R-module $\widetilde{R}/R$ locally.]

(3) Prove that a UFD is normal.

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[1]Also called the **ideal class group** in number theory, particularly for number rings (finite extensions of $\mathbb{Z}$).

(4) Prove that an intersection of normal rings (with fraction field K) is normal. Use Theorem 1 to deduce that if R is Noetherian and normal, then the ring $\bigcap_{ht1P\in SpecR} R_P$ is normal. [In fact: for Noetherian rings, R is normal if and only if $R = \bigcap_{ht1P\in SpecR} R_P$; See Hochster's Dec 4 lecture.]

(5) Let R be a Noetherian normal domain with fraction field K. Define a group homomorphism $K^\times \to \mathrm{Div}R$ so that the divisor class group of R is the cokernel.

(6) Let R be a normal Noetherian domain. Let $\mathfrak{q}$ be $\mathfrak{p}$-primary where $\mathfrak{p}$ is height one.
  (a) Show that $\mathfrak{q} = \mathfrak{p}^{(n)}$ for some n, where by definition, $\mathfrak{p}^{(n)} = \mathfrak{p}^n R_\mathfrak{p} \cap R$.
  [Hint: Recall that $\mathfrak{q}$ is $\mathfrak{p}$-primary if and only if $\sqrt{\mathfrak{q}} = \mathfrak{p}$ and $\mathfrak{q}R_\mathfrak{p} \cap R = \mathfrak{q}$. You will also need Thm 1.]
  (b) Show $\mathrm{div}(f) = n_1[\mathfrak{p}_1] + \cdots + n_t[\mathfrak{p}_t]$ where $\langle f\rangle = \mathfrak{p}_1^{(n_1)} \cap \cdots \cap \mathfrak{p}_t^{(n_t)}$ is a primary decomp.

(7) Dedekind Domains. A **Dedekind domain** is a normal Noetherian ring of dimension 1. Show that the normalization of any finite integral extension of $\mathbb{Z}$ is a Dedekind domain. Such a ring is a called a **number ring.** [**You may assume** that the normalization of a finitely generated $\mathbb{Z}$-algebra is Noetherian. This is a non-trivial fact.[2]]

(8) Let R be a Dedekind domain. Let $\mathfrak{q}$ and $\mathfrak{q}'$ be non-zero primary ideals with distinct radicals.
  (a) Show $\mathfrak{q} \cap \mathfrak{q}' = \mathfrak{q}\mathfrak{q}'$. [Hint: Observe $\mathfrak{q} + \mathfrak{q}' = R$.]
  (b) Explain why the primary decomposition of any ideal in R is *unique.*
  (c) Show $\mathfrak{p}^n \neq \mathfrak{p}^{n+1}$ for all $n \in \mathbb{N}$ and all non-zero $\mathfrak{p} \in \mathrm{Spec}\, R$. [Hint: NAK!]
  (d) Let $\mathfrak{q}$ be $\mathfrak{p}$-primary. Show that $\mathfrak{q} = \mathfrak{p}^n$ for some n. [Hint: Recall that $\mathfrak{q}$ is $\mathfrak{p}$-primary if and only if $\sqrt{\mathfrak{q}} = \mathfrak{p}$ and $\mathfrak{q}R_\mathfrak{p} \cap R = \mathfrak{q}$. You will also need Theorem 1.]
  (e) Prove ideals in a Dedekind domain factor uniquely as a product of prime ideals.

(9) Let $R = K[x, y, z, w]/\langle xy - zw\rangle$.
  (a) Prove that R is a three dimensional domain.
  (b) Prove that R is not a UFD.
  (c) Prove that $\langle x, z\rangle, \langle x, w\rangle, \langle y, z\rangle, \langle y, w\rangle$ are all height one prime ideals.
  (d) Show that $R_{\langle x,z\rangle} \cong K[z, y, \frac{1}{y}, w]_{\langle z\rangle} \cong K[z, y, w]_{\langle z\rangle}$.
  (e) Compute the divisor of $\overline{z} \in R$. [Hint: First find its minimal primes.]
  (f) Show that all primes in (c) are equal in $Cl(R)$ up to sign.
  (g) * Prove that $Cl(R)$ is isomorphic to $\mathbb{Z}$.

(10) Let V be a valuation ring with fraction field V. Prove that V is normal. [Hint: For $\lambda \in K$, consider an equation of integral dependence over V and the possible values of ν for its terms.]

(11) Polynomial Rings over normal rings are Normal.
  (a) Let R be a normal domain. Prove that $R[x]$ is a Normal domain. [Hint: Use the fact that $K[x]$ is a UFD to reduce to considering elements of $K[x]$ integral over $R[x]$.]
  (b) Prove that a directed union of normal domains is normal.
  (c) Prove that a polynomial ring over any normal ring in any number (even infinite) of variables is normal.

(12) * Prove Theorem 1. [The proof of Theorem 2 is in Mel's notes on Dec 4.]

[2] See the paper "Noetherian rings without finite normalization" by Olberding for a detailed discussion.