Worksheet on Primary Decomposition

Note to self: this was hard originally because we should have FIRST proved that for a finitely generated module in a Noetherian ring, the minimal primes of $\operatorname{ann}(M)$ are the same as the minimal associated primes. With out this, they could not prove that Q is P-primary if and only if $\operatorname{Ass}(R/Q) = \{P\}$ in Noetherian rings.

Let R be a commutative ring with 1. Let $\mathfrak{q}$ be an ideal of R.

Definition. A proper ideal $\mathfrak{q}$ is **primary** if $xy \in \mathfrak{q}$ implies that $x \in \mathfrak{q}$ or $y^n \in \mathfrak{q}$ for some $n \in \mathbb{N}$. We say a primary ideal $\mathfrak{q}$ is **$\mathfrak{p}$-primary** if $\sqrt{\mathfrak{q}} = \mathfrak{p}$.

Theorem 1. Let J be an ideal in a Noetherian ring R. Then there exist primary ideals $\mathfrak{q}_1, \dots \mathfrak{q}_t$ such that

$$J = \mathfrak{q}_1 \bigcap \cdots \bigcap \mathfrak{q}_t.$$

Furthermore, the $\mathfrak{q}_i$ can be chosen so that they have distinct radicals $\mathfrak{p}_i$ and the intersection is irredundant. The primary ideals $\mathfrak{q}_i$ are called **primary components** of J, and such an intersection is called a **minimal primary decomposition** of J.

Theorem 2. Let $J = \mathfrak{q}_1 \bigcap \cdots \bigcap \mathfrak{q}_t$ be a minimal primary decomposition of an ideal J in a Noetherian ring R. The set $\{\sqrt{\mathfrak{q}_1}, \sqrt{\mathfrak{q}_2}, \dots, \sqrt{\mathfrak{q}_t}\}$ is precisely $\operatorname{Ass}(R/J)$. In particular, the set of radicals of a primary components of J are uniquely determined (up to order).

Theorem 3. If $\mathfrak{p}$ is a *minimal* associated prime, then the $\mathfrak{p}$-primary component of J is $JR_{\mathfrak{p}} \cap R$. In particular, the *minimal* primary components of J are uniquely determined.

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(1) (a) Show that if $\mathfrak{p}$ is prime, then $\mathfrak{p}$ is primary.
(b) Show that if $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime. So, a primary ideal is $\mathfrak{p}$-primary for some prime $\mathfrak{p}$.
(c) Let $R = \mathbb{C}[x, y, z]$ and $\mathfrak{q} = \langle x^2, xy \rangle$. Show that $\sqrt{\mathfrak{q}}$ is prime but that $\mathfrak{q}$ is not primary. Thus, the converse of (b) fails.

(2) Show that $\mathfrak{q}$ is primary if and only if every zero-divisor of $R/\mathfrak{q}$ is nilpotent.

(3) Primary Decomposition in PIDs.
(a) Prove that and ideal $\mathfrak{q}$ in a PID is primary if and only if it is generated by a power of an irreducible element.
(b) Express the ideal $\langle 12 \rangle \cap \langle 18 \rangle$ of $\mathbb{Z}$ as an intersection of primary ideals.
(c) Is your intersection in (b) irredundant? Are the radicals of your primary components distinct? If not, remove redundant components and/or combine components with the same radical to get a minimal primary decomposition of $\langle 12 \rangle \cap \langle 18 \rangle$.
(d) Find a *minimal* primary decomposition for the ideal $\langle x^4 - x^2 \rangle$ in $K[x]$.

(4) Characterizations of primary. Fix a prime ideal $\mathfrak{p}$ containing an ideal $\mathfrak{q}$. Show that the following are equivalent.

(a) $\mathfrak{q}$ is $\mathfrak{p}$-primary.
(b) The nilradical of $R/\mathfrak{q}$ is $\mathfrak{p}/\mathfrak{q}$ and every zero-divisor of $R/\mathfrak{q}$ is nilpotent.
(c) The radical of $\mathfrak{q}$ is $\mathfrak{p}$ and $\mathfrak{q} = \mathfrak{q}R_\mathfrak{p} \cap R$. [Hint: Show $x \in \mathfrak{q}R_\mathfrak{p} \cap R$ implies $\exists y \notin \mathfrak{p}$ s.t. $xy \in \mathfrak{q}$.]

(5) Show that if $\sqrt{\mathfrak{q}}$ is *maximal*, then $\mathfrak{q}$ is primary. Is the converse true? [Hint: Note $R/\mathfrak{q}$ is local.]

(6) Which of the following ideals in $K[x,y,z]$ is primary: $\langle x^2, y^3, z^5\rangle, \langle xy - z^7\rangle, \langle xy, yz, xz\rangle$.

(7) Let $R = \mathbb{C}[x,y]$. Let $J = \langle x^2, xy\rangle$.
(a) Show that $J = \langle x\rangle \cap \langle x^2, xy, y^n\rangle$ for any $n \geq 1$. [Hint: the UFD property might be useful.]
(b) Show that $J = \langle x\rangle \cap \langle x^2, y - ax\rangle$ for any $a \in \mathbb{C}$.
(c) Show that all decompositions in (a) and (b) are minimal primary decompositions.
(d) Use Theorem 2 to compute $\mathrm{Ass}(R/J)$.
(e) Verify that this example comports with Theorem 3 by computing $JR_\mathfrak{p} \cap R$.

(8) Irreducible ideals. We say that a proper ideal is **irreducible** if it is not the intersection of two strictly larger ideals.
(a) Define $(I : z) := \{y \in R \mid yz \in I\}$. Prove that $(I : z)$ is an ideal (proper iff $z \notin I$).
(b) Prove that $(I : x^n)$ is proper for all $n \in \mathbb{N}$ if and only if $x \notin \sqrt{I}$.
(c) Assume R is Noetherian. Fix $x \in R$. Prove there exists $N \in \mathbb{N}$ such that $(I : x^n) = (I : x^N)$ for all $n \geq N$.
(d) Prove that for any ideal I in a Noetherian ring, and any $x \in R$, $(I + \langle x^n\rangle) \cap (I : x^n) = I$, for $n \gg 0$. [Hint: For $i \in I$, say $i + rx^n \in (I : x^n)$ for $n \geq N$. Use (c) to understand r.]
(e) Prove that an irreducible ideal of a Noetherian ring is primary. [Hint: Say $xy \in I$ but $x \notin \sqrt{I}$.]

(9) Fix a prime ideal $\mathfrak{p}$. Prove that a finite intersection of $\mathfrak{p}$-primary ideals is $\mathfrak{p}$-primary.

(10) Existence of Primary Decomposition. *Assume that R is Noetherian.*
(a) Prove that every ideal of R is an intersection of finitely many irreducible ideals.
(b) Prove Theorem 1. [Hint: Use (8) and (9).]

(11) Let J be a radical ideal in a Noetherian ring.
(a) Prove that J has a primary decomposition whose primary components are the minimal primes of J.
(b) Prove the primary decomposition in (a) is minimal. [Hint: Say $P_1 \cap \cdots \cap P_t \subset Q$. Show $P_i \subset Q$ for some i by otherwise choosing $r_i \in P_i \setminus Q$, and considering $r_1 \ldots r_t \in P_1 \cap \cdots \cap P_t$.]
(c) Prove that each minimal prime of J is in $\mathrm{Ass}(R/J)$. [Hint: Show $P_i = \mathrm{ann}(\overline{r_1} \cdots \widehat{r_i} \cdots \overline{r_t})$ for suitably chosen r_j.]
(d) Prove that if $Q \in \mathrm{Ass}(R/J)$, then $J \subset Q$. [Hint: Write $Q = \mathrm{ann}\,\overline{x}$ for some non-zero $\overline{x} \in R/J$.]
(e) Prove that if $Q \in \mathrm{Ass}(R/J)$, then Q contains some minimal prime of J. [Hint: See hint for (a).]
(f) Prove that if $Q \in \mathrm{Ass}(R/J)$, then Q is a minimal prime of J. [Hint: We can assume $Q = \mathrm{ann}\,\overline{x}$ where $x \notin P_i$. Show $Q \subset P_i$.]
(g) Conclude that for *radical* J, $\mathrm{Ass}(R/J)$ consists of precisely the minimal primes of J.
(h) Verify Theorems 2 and 3 for radical ideals J in a Noetherian ring. [Hint: Intersection commutes with localization.]