Worksheet on Primary Decomposition: Uniqueness

Let R be a commutative ring with 1.

DEFINITION. A proper ideal $\mathfrak{q}$ is **primary** if every zero-divisor in $R/\mathfrak{q}$ is nilpotent.

TERMINOLOGY. A primary ideal $\mathfrak{q}$ always has prime radical $\mathfrak{p}$; we say $\mathfrak{q}$ is **$\mathfrak{p}$-primary**.

THEOREM ON UNIQUENESS OF PRIMARY DECOMPOSITION. Suppose an ideal J in an arbitrary ring admits a primary decomposition

$$J = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_t.$$

Then J admits a **minimal primary decomposition,** meaning that the intersection can be assumed irredundant and that the $\mathfrak{q}_i$ are $\mathfrak{p}_i$-primary for *distinct* primes $\mathfrak{p}_i$. In this case, the set

$$\{\sqrt{\mathfrak{q}}_1, \sqrt{\mathfrak{q}}_2, \dots, \sqrt{\mathfrak{q}}_t\} = \{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_t\}$$

is independent of the choice of minimal primary decomposition. Furthermore, the minimal primes among the set $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_t\}$ are precisely the minimal primes of J, and for these minimal primes, the corresponding primary component is uniquely determined by $\mathfrak{q}_i = JR_{\mathfrak{p}_i} \cap R$.

NOETHERIAN CASE: Every ideal in a Noetherian ring admits a minimal primary decomposition $J = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_t$. In this case, the set $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_t\} = \mathrm{Ass}(R/J)$.

CAUTION: In the non-Noetherian case, primary decompositions do not always exist for a given J, and the radicals of the primary components do not have to be associated primes.

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(1) LEMMA 1. Prove that finite intersection commutes with taking radicals, localization, and computing colons. That is, for any finite set of ideals $J_1, \dots, J_t$ in an arbitrary ring R, prove that
  (a) $\sqrt{(J_1 \cap J_2 \cdots \cap J_t)} = \sqrt{J_1} \cap \sqrt{J_2} \cdots \cap \sqrt{J_t}$;
  (b) $(J_1 \cap J_2 \cdots \cap J_t)U^{-1}R = J_1U^{-1}R \cap J_2U^{-1}R \cdots \cap J_tU^{-1}R$ for any multiplicative set $U \subset R$; and
  (c) $(J_1 \cap J_2 \cap \cdots \cap J_t) : x = (J_1 : x) \cap (J_2 : x) \cap \cdots \cap (J_t : x)$ for arbitrary $x \in R$.

(2) LEMMA 2. Prove that in any ring, if $P_1 \cap P_2 \cap \cdots \cap P_n$ is an intersection of mutually incomparable prime ideals, then minimal primes of this intersection are precisely the P_i. In particular, when is such an intersection prime? [Hint: $P_1P_2 \cdots P_n \subset P_1 \cap P_2 \cap \cdots \cap P_n$.]

(3) MINIMAL PRIMES. Let J be an ideal in an arbitrary ring which admits a primary decomposition $\mathfrak{q}_1 \bigcap \mathfrak{q}_2 \cdots \bigcap \mathfrak{q}_t$.
  (a) By grouping together primary ideals with the same radical, explain why we can assume the $\mathfrak{q}_i$ have distinct radicals $\mathfrak{p}_i$. [Hint: A finite intersection of $\mathfrak{p}$-primary ideals is $\mathfrak{p}$-primary.]
  (b) Prove that $\sqrt{J} = \bigcap_{i=1}^{t} \mathfrak{p}_i$.
  (c) Prove the minimal primes among $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_t\}$ are precisely the min primes of J.
  (d) Observe that (a) and (c) together establish part of the Theorem on Uniqueness of Primary decomposition.

(4) LEMMA 3. Let $\mathfrak{q}$ be any $\mathfrak{p}$-primary ideal in an arbitrary ring R.
   (a) Prove that $\mathfrak{p} \subset \sqrt{(\mathfrak{q} : x)}$. [Hint: First show $\mathfrak{q} \subset (\mathfrak{q} : x)$.]
   (b) Prove that

$$\sqrt{(\mathfrak{q} : x)} \quad = \quad \begin{cases} \mathfrak{p}, & \text{for } x \notin \mathfrak{q} \\ R & \text{for } x \in \mathfrak{q}. \end{cases}$$

   (c) Prove that for $x \notin \mathfrak{p}$, we have $(\mathfrak{q} : x) = \mathfrak{q}$.

(5) UNIQUENESS OF THE $\mathfrak{p}_i$. Let $J = \mathfrak{q}_1 \bigcap \mathfrak{q}_2 \cdots \bigcap \mathfrak{q}_t$ be any primary decomposition of J in which the radicals $\mathfrak{p}_i$ of $\mathfrak{q}_i$ are distinct.
   (a) Use Lemmas 1 and 3 to show that for any $x \in R$,

$$\sqrt{J : x} = \bigcap_{x \notin \mathfrak{q}_i} \mathfrak{p}_i.$$

   (b) Fix i. Explain why, if the decomposition is irredundent, we can find x in every $\mathfrak{q}_j$ *except* $\mathfrak{q}_i$.
   (c) With x as in (b), show that $\sqrt{J : x} = \mathfrak{p}_i$.
   (d) Show that if $\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_t$ and $\mathfrak{q}'_1 \cap \mathfrak{q}'_2 \cap \cdots \cap \mathfrak{q}'_m$ are two different minimal primary decompositions of an ideal J, then

$$\{\sqrt{\mathfrak{q}}_1, \sqrt{\mathfrak{q}}_2, \ldots, \sqrt{\mathfrak{q}}_t\} = \{\sqrt{\mathfrak{q}'_1}, \sqrt{\mathfrak{q}'_2}, \ldots, \sqrt{\mathfrak{q}'_m}\}.$$

   In particular $t = m$. [Hint: Use (a), (c) and Lemma 2.]
   (e) Conclude that part of the Theorem on Uniqueness of Primary decomposition is proven.

(6) MINIMAL COMPONENTS. Let $J = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_t$ be a primary decomposition.
   (a) Let $\mathfrak{q}$ be $\mathfrak{p}$-primary and P be any prime ideal such that $\mathfrak{p} \not\subset P$. Show that $\mathfrak{q}R_P = R_P$.
   (b) Suppose that $\mathfrak{p}_i$ is minimal among $\{\sqrt{\mathfrak{q}}_1, \sqrt{\mathfrak{q}}_2, \ldots, \sqrt{\mathfrak{q}}_t\}$. Prove that $JR_{\mathfrak{p}_i} \cap R = \mathfrak{q}_i$. [Hint: Use Lemma 1(b) and (a).]
   (c) Complete the proof of the Uniqueness Theorem for Primary Decomposition.

(7) Let R be the ring of all sequences of real numbers. Show that R has infinitely many minimal prime ideals, and therefore the zero ideal does not admit a primary decomposition.

(8) LEMMAS ON ASSOCIATED PRIMES. Let M and N be arbitrary R-modules.
   (a) Show $\operatorname{Ass}(M \oplus N) = \operatorname{Ass}(M) \cup \operatorname{Ass}(N)$. [Hint: Consider $0 \to M \to M \oplus N \to N \to 0$.]
   (b) Show that $\operatorname{Ass}(R/J_1 \cap J_2) \subset \operatorname{Ass}(R/J_1) \cup \operatorname{Ass}(R/J_2)$. [Hint: Find $R/J \hookrightarrow R/J_1 \oplus R/J_2$.]
   (c) For all multiplicative sets $U \subset R$,, we have the following inclusion of sets in $\operatorname{Spec} U^{-1}R$: $\{PU^{-1}R \mid P \in \operatorname{Ass}(M) \text{ and } P \cap U = \emptyset\} \subset \operatorname{Ass}(U^{-1}M)$.
   (d) * Prove the converse to (c) when R is Noetherian.
   (e) * For R Noetherian, show that every minimal prime of J is in $\operatorname{Ass}(R/J)$.

(9) Assume that R is Noetherian. Let $J = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ be a minimal primary decomposition.
   (a) Prove that if $\mathfrak{q}$ is $\mathfrak{p}$-primary, then $\operatorname{Ass}(R/\mathfrak{q}) = \{\mathfrak{p}\}$. [Hint: If $Q \in \operatorname{Ass}(R/\mathfrak{q})$, then $Q = (\mathfrak{q} : x)$ for some $x \notin \mathfrak{q}$. Show that $\mathfrak{q} \subset Q \subset \sqrt{\mathfrak{q}}$.]
   (b) Show that $\operatorname{Ass}(R/J) \subset \{\sqrt{\mathfrak{q}_1}, \ldots, \sqrt{\mathfrak{q}_t}\}$. Thus all associated primes contribute to the primary components.
   (c) * Show that every $\sqrt{\mathfrak{q}_i}$ is in $\operatorname{Ass}(R/J)$. [Hint: Take $x \in \mathfrak{q}_i$ for all $i > 1$ but not $\mathfrak{q}_1$. Then $\mathfrak{p}_1 = \sqrt{J : x}$. So $\mathfrak{p}_1$ is the only minimal prime of $J : x$ and hence an associated prime of $J : x$. Note that $R/\mathfrak{p}_1 \hookrightarrow R/(J : x) \hookrightarrow R/J$.]