Worksheet on Projective schemes

Let S be a $\mathbb{N}$-graded ring. Let $S_+ = \bigoplus_{n>0}$ be its irrelevant ideal.

DEFINITION. The space Proj S is the subspace of Spec S consisting of *homogeneous prime ideals* that do not contain S_+ (with its inherited subspace topology from the Zariski topology on Spec S).

THEOREM. The space Proj S has a cover by open sets, each homeomorphic to Spec R for some ring R (usually different R on each open set).

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(1) THE TOPOLOGY ON PROJ S. Fix an $\mathbb{N}$-graded ring S.
   (a) Show that every closed set of Proj S has the form $\mathbb{V}(I) = \{P \in \text{ Proj } S \mid I \subset P\}$ for some *homogeneous* ideal I. [Hint: Recall that I is homogeneous if and only if, whenever $f \in I$, then all graded components of f are in I.]
   (b) Let f be homogeneous in S_+. Show that the set $D_+(f) = \{P \in \text{ Proj } S \mid f \notin P\}$ is open.
   (c) Show that Proj S has a basis by sets of the form $D_+(f)$ where f is homogeneous.

(2) THE COMPLEX PROJECTIVE LINE. Let $S = K[x, y]$ with its standard grading. Consider Proj S with its Zariski topology, and let $\mathcal{X} \subset$ Proj S be the subspace of **closed points.**
   (a) Prove Proj S has exactly one dense point—that is, a point η in every non-empty open set.
   (b) Show that $\mathcal{X} = \text{Proj } S \setminus \{\eta\}$.
   (c) Thinking about Krull dimension, what do you think should be the dimension of Proj S?
   (d) Show that every non-zero homogeneous polynomial in $\mathbb{C}[x, y]$ factors into *linear* polynomials. [Hint: For f of degree d, show that $\frac{f}{y^d} = f(\frac{x}{y}, 1)$.]
   (e) Assume $K = \mathbb{C}$ for the remaining problems. Show that if $P \in \mathcal{X}$, then $P = \langle ay - bx\rangle$ for some $a, b, \in \mathbb{C}$. To what extent are a, b unique?
   (f) Show that $\mathcal{X}$ can be natural identified with the set of lines through the origin in $\mathbb{C}^2$.
   (g) Show that there is a bijection between $\mathcal{X}$ and $\mathbb{C} \cup \infty$ given by sending a line to its slope.
   (h) Show that the Zariski topology on $\mathcal{X}$ is the finite-complement topology.

(3) EASY LEMMAS. Let S be an $\mathbb{N}$-graded ring. Take any homogeneous $f \in S$ of positive degree.
   (a) Show that $S[\frac{1}{f}]$ has a naturally induced $\mathbb{Z}$-grading.
   (b) Recall that $R = [S[\frac{1}{f}]]_0$ is subring of $S[\frac{1}{f}]$. Show that every ideal of R expands to a *homogeneous* ideal of $S[\frac{1}{f}]$. [Hint: The elements of R have the form $\frac{g}{f^t}$ where g is homogeneous.]
   (c) Show that in an arbitrary A-graded ring, if I is homogeneous, so is $\sqrt{I}$.

(4) DEHOMOGENIZATION. Consider $S = K[x, y, z]$ with its standard $\mathbb{N}$-grading.
   (a) Show that $[S[\frac{1}{z}]]_0 = K[\frac{x}{z}, \frac{y}{z}]$ (a polynomial ring in two variables $s = \frac{x}{z}$ and $t = \frac{y}{z}$.)
   (b) Given $f \in S_d$, show that $\frac{f}{z^d} \in [S[\frac{1}{z}]]_0$, and that $\frac{f}{z^d}$ is $f(s, t, 1) \in K[s, t]$.
   (c) For P in Proj S, prove that $PS[\frac{1}{z}] \cap K[\frac{x}{z}, \frac{y}{z}]$ is prime or $z \in P$.
   (d) For each P in Proj S below, compute generators for $PS[\frac{1}{z}] \cap K[\frac{x}{z}, \frac{y}{z}]$ in the ring $K[s, t]$:
      (i) $P = \langle x - az, y - bz\rangle$
      (ii) $P = \langle x - y\rangle$
      (iii) $P = \langle x^2 + y^2 - z^2\rangle$.
      (iv) $P = \langle x, z\rangle$

(e) For each of your answers in (d), sketch the closure in $\operatorname{Spec} K[s,t]$, or in $\operatorname{maxSpec} K[s,t]$, when $K = \overline{K}$. How is this different from sketching the point itself?

(5) PROJECTIVE SPACE $\mathbb{P}^n_K$. Let $S = K[x_0, x_1, \ldots, x_n]$ with its standard grading. Assume $K = \overline{K}$.
 (a) Show that $\bigcup_{i=0}^n D_+(x_i) = \operatorname{Proj} S$.
 (b) Show that each $D_+(x_i)$ is homeomorphic to $\operatorname{Spec} K[\frac{x_0}{x_i}, \ldots, \hat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}]$ (the $\hat{\frac{x_i}{x_i}}$, means that the $i-th$ term is omitted).
 (c) Show point P of Proj S is closed if and only if its height is n. (You don't need $K = \overline{K}$).
 (d) For an arbitrary closed point $P \in \operatorname{Proj} S$ lying in the open chart $D_+(x_0)$, describe P looking "locally" in an appropriate open neighborhood $D_+(x_i)$, identified with $\operatorname{Spec} K[\frac{x_0}{x_i}, \ldots, \hat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}]$.
 (e) Describe the same closed point P as a homogeneous prime in S (by describing generators). Prove that the algebraic set $\mathbb{V}(P)$ in K^{n+1} is a line through the origin.
 (f) Prove that the subspace of closed points of $\mathbb{P}^n_K$ can be identified with the set of lines through $\mathbf{0} \in K^{n+1}$. [Hint: Use 6d.]

(6) PROOF OF THE THEOREM. Let S be an $\mathbb{N}$-graded ring. Take any homogeneous $f \in S$ of positive degree. Set $R = [S[\frac{1}{f}]]_0$.
 (a) Find a continuous map $D_+(f) \to \operatorname{Spec} R$. [Hint: Factor through $\operatorname{Spec} S[\frac{1}{f}]$.]
 (b) Show that the map in (a) is surjective. [Hint: For $P \in \operatorname{Spec} R$, use (3) to show that $\sqrt{PS[\frac{1}{f}]}$ is a homogeneous prime of $S[\frac{1}{f}]$ which contracts to P. You might want to frst try the case where $\deg f = 1$.]
 (c) Show that the map in (a) is one-to-one. [Hint: If $g \in Q \setminus P$, consider $\frac{g^{\deg f}}{f^{\deg g}}$ in R.]
 (d) Show that the map in (a) is an homeomorphism. [Hint: Show $\operatorname{Spec} R \to \operatorname{Spec} S[\frac{1}{f}]$ sending $P \mapsto \sqrt{PS[\frac{1}{f}]}$ is continuous.]
 (e) Conclude that Proj S is covered by open sets each of which is homeomorphic to a topological space of the form $\operatorname{Spec} R$ (for different R). [Hint: Remember $P \in \operatorname{Proj} S$ means $P \not\supset S_+$.]
 (f) If S is finitely generated as an S_0-algebra, prove that Proj S is covered by *finitely many* open sets of the form $\operatorname{Spec} R$.

(7) CONICS. Let $R = K[x,y,z]/\langle x^2 + y^2 - z^2 \rangle$ with its standard $\mathbb{N}$-grading. Let $X = \operatorname{Proj} R$.
 (a) Describe the open sets $D_+(x)$, $D_+(y)$, and $D_+(z)$ algebraically. Sketch pictures geometrically (of the closed points).
 (b) Explain how to see every "conic section" (circles, ellipses, hyperbolas, etc, of various radii/eccentricity as an open chart $D_+(L)$ of X (where L is a linear form) of the space Proj R

(8) PROJECTIVE SPACE AS COMPLETION OF AFFINE SPACE. Let $K = \overline{K}$. Let $S = K[x_0, x_1, \ldots, x_n]$, and denote Proj S by $\mathbb{P}^n_K$.
 (a) For $n = 1, n = 2$, describe how $D_+(x_0) \subset \mathbb{P}^2$ looks as a subset of all lines through the origin in K^{n+1}. What happens when take its closure in $\mathbb{P}^n$? What new lines are we adding?
 (b) Describe the closed points of Proj $S \setminus D_+(x_0)$ both algebraically (giving generators for the homogenous primes) and geometrically (in terms of lines through $\mathbf{0}$.)
 (c) Prove that the closure of $D_+(x_0)$ in $\mathbb{P}^n_K$ is all of $\mathbb{P}^n_K$.
 (d) Prove that $\mathbb{P}^n$ is a disjoint union of $\operatorname{Spec} K[t_1, \ldots, t_n]$ and $\mathbb{P}^{n-1}$. [Hint: Think of t_i as $\frac{x_i}{x_0}$.]

(9) PROJECTIVE CLOSURE. Consider the ring $R = K[s,t]/\langle st - 1 \rangle$.
 (a) Sketch a picture of (the closed points) of $\operatorname{Spec} R$.
 (b) Find homogeneous $F(x,y,z) \in K[x,y,z]$ such that $F(s,t,1) = st - 1$.
 (c) Show that Proj S, where $S = K[x,y,z]/\langle F \rangle$ has an open subset homeomorphic to $\operatorname{Spec} R$.
 (d) Examine Proj S in the charts $D_+(y)$ and $D_+(x)$.
 (e) Describe "the points at infinity" Proj $S \setminus D_+(z)$ and compare to your sketch in (1).