Worksheet on Reviewing Tensor Product

Let R be a commutative ring with 1. Let R-Mod denote the category of R-modules.

UNIVERSAL PROPERTY OF TENSOR PRODUCTS: Any *bilinear* R-module map $M \times N \to Q$ factors uniquely through the universal bilinear map $M \times N \to M \otimes_R N$ sending $(m, n) \mapsto m \otimes n$.

PROPERTIES OF TENSOR. Let A, B, C be R-modules.

(1) IDENTITY: $R \otimes_R A \cong A$ and $A \otimes_R R \cong A$.
(2) ASSOCIATIVITY: $A \otimes_R (B \otimes_R C) \cong (A \otimes_R B) \otimes_R C$.
(3) DISTRIBUTIVITY: $A \otimes_R (B \oplus_R C) \cong (A \otimes_R B) \oplus_R (A \otimes_R C)$.
(4) COMMUTATIVITY: $A \otimes_R B \cong B \otimes_R A$.

DEFINITION. A covariant[1] functor Γ from $R-\textbf{Mod}$ to $S-\textbf{Mod}$ is **right exact** if it preserves cokernels—that is, if for every exact sequence of R-modules

$$M_1 \to M_2 \to M_3 \to 0,$$

the sequence

$$\Gamma(M_1) \to \Gamma(M_2) \to \Gamma(M_3) \to 0.$$

Similarly, it is left exact if it preserves kernels (meaning that if $0 \to M_1 \to M_2 \to M_3$ is exact, then so is $0 \to \Gamma(M_1) \to \Gamma(M_2) \to \Gamma(M_3)$.) The functor Γ is **exact** if it is both left and right exact.

DEFINITION: An R-module M is **flat** if the functor $N \mapsto M \otimes_R N$ from R-mod to R-mod is exact.

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(1) TENSOR PRODUCTS OF VECTOR SPACES. Let M and N be finite dimensional vector spaces over a field K of dimensions m, n respectively.
  (a) Fix bases so that $M \cong K^m$ and $N \cong K^n$ are identified with spaces of column vectors in the usual way. Using the matrix multiplication map
  $$K^m \times K^n \to K^{m\times n} \qquad (\vec{v}, \vec{w}) \mapsto \vec{v} \cdot \vec{w}^{tr},$$
  prove that $K^m \otimes_K K^n$ is naturally[2] isomorphic to the space $K^{m\times n}$ of $m \times n$ matrices.
  (b) Conclude that $M \otimes_K N$ has dimension mn, and describe an explicit basis in terms of bases for M and N.
  (c) Let $X \subset K^{m\times n}$ be the image of the bilinear map in (a). Explain why X consists of the matrices of rank at most 1, and why this is an algebraic set.
  (d) How likely is it that a randomly chosen element of $M \otimes_K N$ can be written as $m \otimes n$?

(2) Explain why the tensor product of finitely generated modules is finitely generated, and why the tensor product of free modules (of rank m and n respectively) is free (of rank mn).

(3) RIGHT EXACTNESS OF TENSOR. For fixed M, consider the covariant functor $M \otimes_R -$ from R-mod to R-mod given by $N \mapsto M \otimes_R N$.

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[1]Left and right exactness are defined similarly for contravariant functors.
[2]meaning that the isomorphism can be described without chosing a basis.

(a) Prove that $M \otimes_R -$ is right exact. [Hint: Find a surjection $M \otimes_R M_2 / \operatorname{im}(M \otimes_R M_1) \to M \otimes_R M_3$. To show it is an isomorphism, construct a bilinear map $M \times M_3 \to M \otimes_R M_2 / \operatorname{im}(M \otimes_R M_1)$. Be sure to check your bilinear map is well-defined.]

(b) Prove that $M \otimes_R -$ is not left exact in general. [Hint: Take $M = R/I$ and consider a sequence $0 \to R \xrightarrow{f} R \to R/\langle f \rangle$ where $f \in I$ is a non-zero-divisor of R.]s

(c) Prove free modules are flat. Conclude that $V \otimes_K -$ is exact in the category K-Vector spaces, for any V.

(4) Localization. Let $U \subset R$ be a multiplicative system. For an R-module M, define $U^{-1}M$ as the set of equivalence classes $\frac{m}{u}$ where $m \in M$ and $u \in U$ with $\frac{m}{u} \sim \frac{m'}{u'}$ if there exists $v \in U$ such that $v(u'm - um') = 0$.

(a) Explain why $M \mapsto U^{-1}M$ is a functor from R-mod to $U^{-1}R$-mod.

(b) For $R = \mathbb{Z}$ and $U = \mathbb{Z} \setminus \{0\}$, show that the localization functor kills torsion modules.

(c) Prove there is a unique isomorphism $U^{-1}R \otimes_R M \cong U^{-1}M$. [Hint: Universal properties.]

(d) Prove that the localization functor is exact.

(e) Show that $U^{-1}R$ is a flat R-module.

(5) Base Change. Let $R \to S$ be a ring homomorphism.

(a) Show that $M \mapsto S \otimes_R M$ is a right exact functor from R-mod to S-mod.

(b) In particular, show that $S \otimes_R R/I \cong S/IS$ for all ideals $I \subset R$ and $S \otimes_R R[x_1, \dots, x_n]/J \cong S[x_1, \dots, x_n]/JS[x_1, \dots, x_n]$ for all ideals $J \subset R[x_1, \dots, x_n]$.

(c) Show that this functor is not left exact in general. [Hint: Use (5).]

(d) For any multiplicative system $U \subset R$, show that the functor from R-mod to $U^{-1}R$-mod sending $M \mapsto U^{-1}R \otimes M$ is exact.

(e) Show that if S is a flat R-algebra, then $S \otimes_R I \cong IS$ for all ideals $I \subset R$.

(6) Adjointness of Tensor and Hom. Let $R \to S$ be a ring homomorphism. Let M and N be S modules, and let Q be an R-module.

(a) Discuss natural R-module structures on M and N.

(b) Discuss a natural S-module structure on $\operatorname{Hom}_R(N, Q)$.

(c) Given a R-bilinear map $M \times N \to Q$, describe a natural R-module map $M \to \operatorname{Hom}_R(N, Q)$.

(d) Show that there is an R-module isomorphism

$$\operatorname{Hom}_R(M \otimes_R N, Q) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_R(N, Q)).$$

(e) Show that there is an S-module isomorphism

$$\operatorname{Hom}_R(M \otimes_S N, Q) \cong \operatorname{Hom}_S(M, \operatorname{Hom}_R(N, Q)).$$

(7) Tensor Product of Algebras. Let A and B be R-algebras.

(a) Show that $A \otimes_R B$ has the structure of an R-algebra.

(b) Show that there are R-algebra maps $A \to A \otimes_R B$ and $B \to A \otimes_R B$ which make $A \otimes_R B$ into a **coproduct** in the category of R-algebras. [Meaning: given any R-algebra T to which both A and B map, $\exists!$ R-algebra map $A \otimes_R B \to T$ making the relevant diagrams commute.]

(8) Let K be an algebraically closed field, and let $V \subset K^n$ and $W \subset K^m$ be algebraic sets.

(a) Show that $V \times W$ is an algebraic set in K^{n+m}.

(b) * Show that its coordinate ring is isomorphic to $K[V] \otimes_K K[W]$.