Worksheet on the Zariski topology and the Functor *Spec*

Let R be any ring. In this course, "ring" means *commutative ring with unity.*

DEFINITION: The **spectrum** of R, denoted $\operatorname{Spec} R$, is the set of all prime ideals of R.
NOTATION: For any subset $\mathcal{S} \subset R$, define $\mathbb{V}(\mathcal{S}) := \{P \in \operatorname{Spec} R \mid \mathcal{S} \subset P\}$.

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(1) a) Describe Spec k, when k is a field.
b) For arbitrary R, compute $\mathbb{V}(\{0\})$ and $\mathbb{V}(\{1\})$.
c) Explain why Spec R has a natural structure of a poset. Describe this poset for $\operatorname{Spec} \mathbb{Z}$.
d) Describe $\mathbb{V}(\{24\})$ in Spec $\mathbb{Z}$.

(2) THE ZARISKI TOPOLOGY
a) Prove that $\mathbb{V}(\mathcal{S}) \cap \mathbb{V}(\mathcal{S}') = \mathbb{V}(\mathcal{S} \cup \mathcal{S}')$.
b) Prove that $\mathbb{V}(\mathcal{S}) \cup \mathbb{V}(\mathcal{S}') = \mathbb{V}(\mathcal{S}\mathcal{S}')$, where $\mathcal{S}\mathcal{S}'$ denotes the subset $\{fg \mid f \in \mathcal{S},\, g \in \mathcal{S}'\}$.
c) Prove that $\operatorname{Spec} R$ has the structure of a topological space, whose closed sets are those of the form $\mathbb{V}(\mathcal{S})$ for some subset $\mathcal{S}$. [Hint: First, rephrase the definition of topology in terms of *closed sets* instead of open sets.] This is the **Zariski topology** on $\operatorname{Spec} R$.

(3) Let k be an algebraically closed field. Consider the topological space Spec $k[x]$.
a) Describe the points. (There are two types)
b) Find a dense point. Is this space Hausdorff?
c) Show that $\mathbb{V}(\mathcal{S}) = \mathbb{V}(\{f\})$ where f is the greatest common divisor of all the elements of $\mathcal{S}$.
d) Let maxSpec $k[x]$ be the subset consisting of maximal ideals. Prove that the subspace topology on maxSpec $k[x]$ is the finite complement topology.

(4) Recall: the **radical** of an ideal I of R is the ideal $\sqrt{I} := \{f \in R \mid f^n \in I \text{ for some } n \in \mathbb{N}\}$.
a) Show that $\mathbb{V}(\mathcal{S}) = \mathbb{V}(\langle \mathcal{S} \rangle)$, where $\langle \mathcal{S} \rangle$ denotes the ideal generated by $\mathcal{S}$.
b) Prove that $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$ for any ideal $I \subset R$.

(5) Consider the topological space Spec $\mathbb{C}[x, y]$.
a) Fix a point $= (a, b) \in \mathbb{C}^2$. Use the "evaluation at p" map $\mathbb{C}[x, y] \xrightarrow{e_p} \mathbb{C}$ to prove that the set of polynomial functions vanishing at p is a maximal ideal m_p of $\mathbb{C}[x, y]$.
b) Find generators for m_p. Can this ideal be principal? [Hint: use the Math 593 fact that $\mathbb{C}[x, y]$ is a UFD.]
c) Show that the map $\eta : \mathbb{C}^2 \to \operatorname{Spec} \mathbb{C}[x, y]$ sending $p = (a, b)$ to m_p is injective.
d) Find at least two distinct points of $\operatorname{Spec} \mathbb{C}[x, y]$ not in the image of η.
e*) Describe the closed sets of $\operatorname{Spec} \mathbb{C}[x, y]$. There are four types.

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(6) Let $R \xrightarrow{\phi} S$ be a ring homomorphism.
a) Prove that if $P \in \operatorname{Spec} S$, then $\phi^{-1}(P) \in \operatorname{Spec} R$.

b) Prove that the map

$$\operatorname{Spec} S \xrightarrow{\phi^{\#}} \operatorname{Spec} R \quad P \mapsto \phi^{-1}(P)$$

is continuous in the Zariski topology.

c) For the ring map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ sending each integer to its residue class modulo 2, describe the induced map on Spectra explicitly.

d) For the ring map $\mathbb{Z} \hookrightarrow \mathbb{Q}$, describe the the induced map on Spectra explicitly.

(7) The Functor Spec. Consider the mapping

$$\mathbf{commRing} \to \mathbf{Top} \qquad R \mapsto \operatorname{Spec} R$$

from the category of commutative rings with identity to the category of topological spaces.

a) Explain using (6) how this mapping defines a contravariant functor[1] from **commRing** to **Top**. This is the **functor "Spec"**.

b) Do isomorphic rings have Spectra that are homeomorphic topological spaces? Explain.

c) If two rings have homeomorphic spectra, are they necessarily isomorphic rings? Explain.

(8) Find a homeomorphism between $\operatorname{Spec} \mathbb{C}[x]$ and $\operatorname{Spec} \mathbb{C}[x,y]/\langle x-y\rangle$.

(9) Closed Embeddings. Consider the natural quotient map $R \overset{\phi}{\to} R/I$, where $I \subset R$ is any ideal. Prove that the corresponding map on Spectra can be identified with the inclusion of the closed set $\mathbb{V}(I)$ (with its subspace topology) in the space $\operatorname{Spec} R$.

(10) Nilradicals and Reduced Rings. Recall that for any commutative ring R, the **nilradical** is the ideal $\sqrt{\langle 0\rangle}$ of all nilpotent elements, and the quotient $R_{red} = R/N$ is called the **reduced ring of** R.

a) Prove that $\mathbb{V}(N) = \operatorname{Spec} R$.

b) Prove that natural quotient map $R \to R_{red}$ induces a homeomorphism $\operatorname{Spec} R_{red} \to \operatorname{Spec} R$.

c) Describe Spec R where $R = \mathbb{Z}[x,y]/\langle 24, x^4, y^6\rangle$.

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(11) Basic Open Sets. For $f \in R$, define $D(f) := \{P \in \operatorname{Spec} R \mid f \notin P\}$.

a) Prove that $D(f)$ is an open set of $\operatorname{Spec} S$.

b) Prove that the open sets of the form $D(f)$ form a **basis** for the Zariski topology on $\operatorname{Spec} R$.

c) For any ideal $I \subset R$, prove that $\mathbb{V}(I) = \emptyset$ if and only if $1 \in I$.

(12) Quasi-compactness of the Zariski topology.

a) Prove that $\operatorname{Spec} R$ is quasi-compact (meaning that every open cover has a finite subcover). [Hint: Use (c).]

b) Prove that if R is Noetherian, then every open set of $\operatorname{Spec} R$ is quasi-compact.

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[1]Recall that a functor is a mapping for both objects and for morphisms which satisfies two axioms: the functor should preserve identity maps, and respect composition.