Worksheet on Valuation Rings

Fix a field K.

DEFINITION: A **valuation ring** of a field K is a subring V with the property that for all non-zero $x \in K$, we have $x \in V$ or $x^{-1} \in V$.

DEFINITION: An abelian group $(\Gamma, +)$ is **ordered** if there is some total ordering $>$ on the elements of Γ which satisfies the axiom: $x > y$ and $z > w$ implies $x + z > y + w$.

DEFINITION: A **valuation** (or Γ-valued valuation) on K is a group homomorphism $K^\times \xrightarrow{\nu} \Gamma$, where Γ is an ordered abelian group, which satisfies the axiom:

$$\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$$

for all $x, y \in K^\times$. We make the convention that $\nu(0) = \infty$, which is larger than any element of Γ.

If $k \subset K$ is a subfield, we say that ν is a k**-valuation** if $\nu(\lambda) = 0$ for all $\lambda \in k^\times$.

DEFINITION: The **value group** of a valuation $K^\times \xrightarrow{\nu} \Gamma$ is in the *image* of ν in Γ. We say a valuation is **discrete** if the value group is isomorphic to $\mathbb{Z}$.

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(1) p-ADIC VALUATION RINGS. Show that $\mathbb{Z}_{\langle p \rangle}$ is a valuation ring.

(2) Which of the following sets is a *valuation ring* of the field $\mathbb{C}(x, y)$.
  (a) The subring $R = \mathbb{C}[x, y]_{\langle y \rangle}$. [Hint: Use the UFD property of $\mathbb{C}[x, y]$ to characterize elements of R.]
  (b) The subring $\mathbb{C}[x, y]_{\langle x,y \rangle}$. [Hint: Consider $\frac{x}{y}$.]
  (c) Let S be the subset of rational functions $f(x, y)/g(x, y)$ where order $f \geq$ order g. Here, the **order** $f \in \mathbb{C}[x, y]$ is the degree of the *lowest* degree non-zero term in f (or ∞ if $f = 0$).

(3) Prove that every ordered abelian group is torsion free. [Hint: Reduce to the case $x > 0$.]

(4) Let ν be an arbitrary valuation on K. Say $f, g \in K^\times$.
  (a) Show that $\nu(\frac{1}{f}) = -\nu(f)$.
  (b) Show that $\nu(f) = \nu(-f)$. [Hint: Use (3).]
  (c) Show that if $\nu(f) > \nu(g)$, then $\nu(f + g) = \nu(g)$. [Hint: $(f + g) + (-f) = g$.]

(5) Consider the field $\mathbb{C}(x, y)$.
  (a) Show that there is a *unique* $\mathbb{R}$-valued $\mathbb{C}$-valuation ν_π of $\mathbb{C}(x, y)$ such that $\nu_\pi(x^a y^b) = a + \pi b$. What is the value group of ν_π? [Hint: Use (4) (a) and (c).]
  (b) Consider $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ as an ordered group with the lexicographical order, so that $(a, b) > (c, d)$ if $a > 0$ *or* $a = c$ and $b > d$. Show that there is a *unique* Γ-valued $\mathbb{C}$-valuation ν of $\mathbb{C}(x, y)$ such that $\nu(x^a y^b) = (a, b)$. [Hint: Use (4).]
  (c) Are either of the valuations constructed in (a) or (b) discrete?

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(6) VALUATIONS DETERMINE VALUATION RINGS. Let ν be an arbitrary valuation on K.
  (a) Prove that the set $V \subset K$ of elements f such that $\nu(f) \geq 0$ is a valuation ring of K. We call this the **valuation ring** of ν.
  (b) Show that f is a unit in V if and only if $\nu(f) = 0$.
  (c) Show that V is local with maximal ideal $\{f \in K \mid \nu(f) > 0\}$.
  (d) Show that $\forall \gamma \in \Gamma$, the sets $\{f \in K \mid \nu(f) > \gamma\}$ and $\{f \in K \mid \nu(f) \geq \gamma\}$ are ideals of V.
  (e) Show that if $k \subset K$ is a subfield such that $\nu_{|k^\times} = 0$, then V is a k-algebra.

(7) MORE p-ADIC VALUATIONS. Fix a prime number p.
  (a) Using the UFD property of $\mathbb{Z}$, explain why every non-zero element a/b of $\mathbb{Q}$ can be written (up to sign and reordering) as $p_1^{a_1} \dots p_t^{a_t}$ where the p_i are distinct primes and $a_i \in \mathbb{Z}$.
  (b) For non-zero $a/b \in \mathbb{Q}$, define $\nu_p(a/b) = a_p$, the exponent of p where a/b is written as in (a). Show that ν_p is a discrete valuation on $\mathbb{Q}$.
  (c) Show that the valuation ring of ν_p is $\mathbb{Z}_{\langle p \rangle}$.

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(8) IDEALS IN VALUATION RINGS. Let V be a valuation ring of K.
  (a) Show that for any two non-zero elements $x, y \in V$, either $x|y$ or $y|x$.
  (b) Show that every finitely generated ideal of V is *principal*.[1]
  (c) Prove that the ideals of V form a totally ordered set. That is, for any two ideals I, J in V, either $I \subset J$ or $J \subset I$. [Hint: If $x \in I$, $x \notin J$, consider whether $y \in J$ can divide x.]
  (d) Prove that valuation rings are local.

(9) Explain why a *Noetherian* valuation ring is a local PID. Compute its Krull dimension. [Use (8).]

(10) Let $\nu : K^\times \to \mathbb{Z}$ be a discrete valuation on K, and let (V, m) be the corresponding **discrete valuation ring.**
  (a) Let $I \subset V$ be any non-zero ideal, explain why we can chose an element $f \in I$ such that $\nu(f)$ is minimal (among elements in I).
  (b) Prove that every ideal of V is principal. In particular, a discrete valuation ring is Noetherian.
  (c) Let t generate m. Show that all other ideals in V are generated by t^n for some $n \in \mathbb{N}$.

(11) For the $\mathbb{R}$-valued valuation ν_π on $\mathbb{C}(x, y)$ constructed in (5a), let (V, m) be the valuation ring.
  (a) Prove that m is not finitely generated. [Hint: for any positive real number ϵ, we can always find $a, b \in \mathbb{Z}$ such that $0 < a + b\pi < \epsilon$.]
  (b) Prove that m is the only non-zero radical ideal. [Hint: If I is radical, take $f \in I$. Now take arbitrary $h \in m$ and shown that $h^n \in \langle f \rangle \subset I$ for $n \gg 0$.]
  (c) Prove that V is a non-Noetherian ring of Krull dimension one.

(12) Consider the $\mathbb{Z} \oplus \mathbb{Z}$-valued valuation ν on $\mathbb{C}(x, y)$ from (5b). Let (V, m) be its valuation ring.
  (a) Show that $m := \{f \in K \mid \nu(f) \geq (0, 1)\}$, and that this ideal is generated by y.
  (b) Show that the ideal $\langle x \rangle$ of V is not prime. [Hint: $x = xy^{-1} \cdot y$.]
  (c) Show that the set $P := \{f \in V \mid \nu(f) = (a, b) \text{ where } a \geq 1\}$ is a prime, non-maximal ideal of V.
  (d) Show that $\dim V \geq 2$. Even more (*): show $\dim V = 2$.
  (e) Construct an infinite ascending chain of ideals of V. [Hint: $\langle xy^{-1} \rangle \subset \langle xy^{-2} \rangle \subset \dots$]

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(13) VALUATION RINGS DETERMINE VALUATIONS. Let V be a valuation ring of K (with $V \neq K$).
  (a) Let Γ' be the set of non-zero principal ideals of V. Show that Γ' has the structure of a totally ordered abelian monoid (under multiplication). What is the identity element?
  (b) Explain how to extend Γ' to an ordered abelian group Γ whose elements are the rank one free V-submodules of K.
  (c) Construct a valuation $\nu : K^\times \to \Gamma$ whose valuation ring recovers V.
  (d) Show that if V is Noetherian, then the corresponding valuation is **discrete**–that is, $\Gamma \cong \mathbb{Z}$.

[1]Likely, your argument also shows that any ideal I that contains an element f such that $\nu(f)$ is minimal is principal.