Worksheet on Algebraic Sets

Fix a field K.

(1) For a point $p = (a_1, \dots, a_n) \in K^n$, consider the "evaluation at p" map $K[x_1, \dots, x_n] \xrightarrow{e_p} K$ sending $f \mapsto f(p)$.
  (a) Show that e_p is a ring homomorphism and find generators for its kernel, m_p.
  (b) Show that evaluation gives a bijection between K^n and the subset of $\operatorname{Spec} K[x_1, \dots, x_n]$ consisting of maximal ideals whose residue field is isomorphic to K.
  (c) Find a maximal ideal of $\mathbb{R}[x, y]$ not of the form m_p for any $p \in \mathbb{R}^2$.
  (d) Use Hilbert's Nullstellensatz (below) to describe the points of $\operatorname{maxSpec} \mathbb{C}[x_1, \dots, x_n]/I$.

Hilbert's Nullenstellensatz: If K is algebraically closed, then *every* maximal ideal of $K[x_1, \dots, x_n]$ is of the form m_p for some $p \in K^n$.

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(2) Algebraic Sets. Let $\mathcal{S}$ be any collection of polynomials in $K[x_1, \dots, x_n]$. Define the corresponding **algebraic set** in K^n as

$$\mathbb{V}(\mathcal{S}) := \{p \in K^n \mid f(p) \ \forall f \in \mathcal{S}\}.$$

  (a) Prove $\mathbb{V}$ is order-reversing: if $\mathcal{S} \subset \mathcal{S}'$, then $\mathbb{V}(\mathcal{S}') \subset \mathbb{V}(\mathcal{S})$.
  (b) Show that $\mathbb{V}(\mathcal{S}) = \mathbb{V}(\langle \mathcal{S} \rangle) = \mathbb{V}(\sqrt{\langle \mathcal{S} \rangle})$.
  (c) Prove that every algebraic set can be defined by *finitely many* polynomials.
  (d) Describe each algebraic set in the cases below:
    i). $\mathcal{S} = \{x^2 - y\} \subset \mathbb{R}[x, y]$.
    ii). $\mathcal{S} = \{n(x^2 + y^2 - z^2) \mid n \in \mathbb{N}\} \subset \mathbb{R}[x, y, z]$.
    iii). For K arbitrary, $\mathcal{S}$ any collection of *linear* polynomials in n variables.
    iv). $\mathcal{S} = \{x^2 + 1\} \subset \mathbb{R}[x]$.
    v). $\mathcal{S} = \{x^2 + 1\} \subset \mathbb{C}[x]$.
    vi). $\mathcal{S} = \langle f \rangle$, where f is an arbitrary non-zero, non-constant in $\mathbb{C}[x]$.
    vii). K algebraically closed, $\mathcal{S} \subset K[x]$ arbitrary. (There are three cases).
  (e) Explain why $SL_n(n, K)$ has the structure of an algebraic set.
  (f) Explain why the set of $m \times n$ matrices of rank $< t$ is an algebraic set (contained in $K^{m \times n}$).

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(3) Let $V \subset K^n$ be an algebraic set. Define an ideal in $K[x_1, \dots, x_n]$ by

$$\mathbb{I}(V) := \{f \in K[x_1, \dots, x_n] \mid f(p) = 0 \ \forall p \in V\}.$$

  (a) Show that $\mathbb{I}(V)$ is a radical ideal in $K[x_1, \dots, x_n]$.
  (b) For $V = \mathbb{V}(\mathcal{S})$, show that $\langle \mathcal{S} \rangle \subset \sqrt{\langle \mathcal{S} \rangle} \subset \mathbb{I}(V)$.
  (c) For $V = \mathbb{V}(\mathcal{S})$, show that $\mathbb{V}(\mathcal{S}) = \mathbb{V}(\mathbb{I}(V))$.
  (d) Give an example (when $K = \mathbb{R}$, say) to show that $\sqrt{\langle \mathcal{S} \rangle} \subset \mathbb{I}(V)$ can be *proper.*
  (e) When K is algebraically closed, use the Nullstellensatz (below) to find explicit bijections between algebraic sets in K^n and radical ideals in $K[x_1, \dots, x_n]$.

Hilbert's Nullenstellensatz: If K is algebraically closed, then for all algebraic sets $V \subset K^n$, $\mathbb{I}(V) = \sqrt{\langle \mathcal{S} \rangle}$ as ideals in $K[x_1, \dots, x_d]$.

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(4) Coordinate Rings. Let $V \subset K^n$ be an algebraic set. Let $\mathcal{R}$ be the ring of *all K-valued functions* on the set V. Consider the restriction map $\rho : K[x_1, \dots, x_n] \to \mathcal{R}$ sending $f \mapsto f_{|V}$.

(a) Verify that ρ is a ring homomorphism and compute its kernel. The image is called the **coordinate ring** of V and denoted $K[V]$.

(b) Prove that $\mathcal{R}$, and hence $K[V]$, is *reduced.*

(c) Give a K-algebra presentation for $K[V]$. Explain why $K[V]$ is called the **coordinate ring**. [Note: the "$i-th$ coordinate function" on K^n is the function x_i which takes each point $p = (a_1, \dots, a_n) \in V$ to its $i-th$ coordinate a_i.]

(d) Describe the coordinate ring for the following algebraic sets:

(i) $\mathbb{V}(y - x^2) \subset K^2$.

(ii) $\mathbb{V}(y^2, xy, x^{17}) \subset K^2$.

(iii) * The set of 3×3 matrices over $\mathbb{F}_5$ of determinant $1_{\mathbb{F}_5}$ and trace $0_{\mathbb{F}_5}$.

(5) Let R be a finitely generated K algebra. Fix a presentation $R \cong K[x_1, \dots, x_n]/I$. Describe natural bijections between the following three sets

- $\mathbb{V}(I) \subset K^n$
- Maximal ideals of R with residue field isomorphic to K.
- $Hom_{K-alg}(R, K)$, the set of morphisms from R to K in the category of K-algebras.

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(6) Fix an algebraic set V in K^n. Define the **Zariski topology** on V by letting the closed sets be *sub-algebraic sets.*

(a) Make sure you can prove this really defines a topology.

(b) Show $D(f) = \{p \in V \mid f(p) \neq 0\} \subset V$ (where f ranges through $f \in K[V]$) form a basis.

(c) Show that the map $V \to \operatorname{maxSpec} k[V]$ sending a point p to m_p is a homeomorphism.

(d) The Zariski topology on V can be identified with the subspace of *closed points* in $\operatorname{Spec} K[V]$. Why?

(e) Is the Zariski topology on V Hausdorff?

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(7) Let $f_1, \dots, f_t$ be polynomials in n variables. Consider the map $\Phi : K^n \to K^t$ given by $p \mapsto (f_1(p), \dots, f_t(p))$.

(a) Show that there is an induced map between coordinate rings $\phi : K[K^t] \to K[K^n]$ sending $g \to g \circ \Phi$. Describe ϕ explicitly in terms of the coordinate-function generators for $K[K^t]$.

(b) For algebraic sets $V \subset K^n$ and $W \subset K^t$, show that $\Phi(V) \subset W$ if and only if $\phi(\mathbb{I}(W)) \subset \mathbb{I}(V)$.

(c) Show that Φ restricts to a map $V \xrightarrow{\Phi_{|V}} W$ if and only if the map $K[K^t] \xrightarrow{\Phi} K[K^n] \xrightarrow{\rho} K[V]$ factors through $K[W]$. [Here, ρ is the restriction map.]

(d) For each of the given maps of coordinate rings, describe the corresponding map of algebraic sets. Compare to the corresponding maps of Spec.

- $\mathbb{C}[x, y] \to \mathbb{C}[t]$ with $x \mapsto t, y \mapsto t$.
- $\mathbb{C}[t] \to \mathbb{C}[x, y]$ with $t \mapsto x$.
- $\mathbb{C}[t] \hookrightarrow \mathbb{C}[t, x, y]/\langle xy - t\rangle$.

(8) * Show that the image of the map $K \to K^3$ sending $t \mapsto (t, t^2, t^3)$ is an algebraic set defined by **two** polynomials. Is the image of *every* polynomial map $K^n \to K^t$ an algebraic set?

(9) * Let ζ_n be an n-th root of unity and let the cyclic group $C_n = \langle \zeta_n \rangle$ act $\mathbb{C}$-linearly on $\mathbb{C}[x, y]$ by $x \mapsto \zeta_n x$, $y \mapsto \zeta_n y$. Show that the ring of invariant polynomials $R = \{f \in \mathbb{C}[x, y] \mid g \cdot f = f \ \forall g \in C_n\}$ is finitely generated and show that $\operatorname{maxSpec} R$ can be identified with the set of orbits for the induced action of C_n on K^2.