Math 156 Applied Honors Calculus II Final Exam Review Sheet Fall 2024

For full credit, justify your answer, and give the units if appropriate. You may use the antiderivative $\int \sec^3\theta d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta))$.

1. **True or false?** Justify your answer with a reason or counterexample.

a) $\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \cdots + \frac{n}{n} = \frac{n+1}{2}$

b) If $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$, then $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f'(x_i)\Delta x = f(b) - f(a)$.

c) If the integral $\int_a^b f(x)\,dx$ is approximated by the right-hand Riemann sum and the number of intervals n is doubled, then the error decreases by approximately 1/4.

d) If $f(0) = f(1) = g(0) = g(1) = 0$, then $\int_0^1 f(x)g''(x)dx = \int_0^1 f''(x)g(x)dx$.

e) The area of the ellipse defined by $4x^2 + 9y^2 = 36$ in the xy-plane is 6π.

f) $\int_0^\infty \frac{dx}{x^2}$ is a convergent improper integral.

g) A spring has natural length 20 cm. If 2 Joules of work are needed to stretch the spring from length 20 cm to 30 cm, then 4 Joules of work are needed to stretch it from length 30 cm to 40 cm.

h) The surface area obtained by rotating the curve $y = \sqrt{x}$, $0 \le x \le 1$, about the x-axis is $\frac{\pi}{6}(5\sqrt{5} - 1)$.

i) The center of mass of the region $\{(x,y) : -1 \le x \le 1, 0 \le y \le \cosh x\}$ is $(\overline{x}, \overline{y}) = (0, \frac{1}{2})$.

j) If $f(x)$ is the pdf of a random variable with mean μ, then $f(x)$ has its maximum value at $x = \mu$.

k) If $f(x)$ is the pdf of a random variable with mean μ, then $\int_{-\infty}^{\infty}(x-\mu)f(x)dx = 0$.

l) If a radioactive material has a half-life of 100 years and a sample has mass 1 kg, then there will be 0.25 kg remaining after 400 years.

m) If \$1000 is invested at 5% annual interest rate and the investment is compounded continuously, then after 2 years the investment is worth between \$1105 and \$1112.

n) $y(t) = 0$ is a stable constant solution of the differential equation $y' = y(1 - y^2)$.

o) If a differential equation $y' = f(y)$ has a unique constant solution $y_1(t) = c$, and $y_2(t)$ is any other solution with initial condition $y_2(0) \ne c$, then $\lim\limits_{t\to\infty} y_2(t) = c$.

p) If a differential equation $y' = f(y)$ is solved by Euler's method, and the step size Δt decreases by a factor of $\frac{1}{2}$, then the error in the numerical solution increases by a factor of approximately $\frac{1}{2}$.

q) If $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = \infty$, then $\lim\limits_{n\to\infty} a_n b_n = 0$.

r) If $0 \le a_n \le b_n$ and $\sum\limits_{n=1}^{\infty} a_n$ converges, then $\sum\limits_{n=1}^{\infty} b_n$ also converges.

s) $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} < \int_1^\infty \frac{dx}{x^2}$ t) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = 0$

u) $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum\limits_{n-1}^{\infty} (\frac{1}{n} - \frac{1}{n+1}) = \sum\limits_{n=1}^{\infty} \frac{1}{n} - \sum\limits_{n=1}^{\infty} \frac{1}{n+1} = \infty - \infty = 0$

v) The alternating series test can be used to show that $\sum\limits_{n=0}^{\infty} (-1)^n$ diverges.

w) The ratio test can be used to show that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges.

x) If the power series $\sum\limits_{n=0}^{\infty} c_n x^n$ converges for $x = 1$, then it also converges for $x = -1$.

y1) If the power series $\sum\limits_{n=0}^{\infty} c_n (x-1)^n$ converges for $x = 2$, then it also converges for $x = \frac{1}{2}$.

y2) The series $\sum\limits_{n=0}^{\infty} 2^n (x-1)^n$ converges for $\frac{1}{2} \le x \le \frac{3}{2}$.

z1) $\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^n (n+1)x^n$ for $|x| < 1$ z2) If $f(x) = e^{-x^2}$, then $f^{(3)}(0) = 0, f^{(6)}(0) = -6$.

z3) $2 < e < 3$ aa) $\int_0^1 e^{-x^2} dx > \frac{2}{3}$ bb) $\frac{\pi}{2} - \frac{1}{3!}(\frac{\pi}{2})^3 + \frac{1}{5!}(\frac{\pi}{2})^5 - \frac{1}{7!}(\frac{\pi}{2})^7 + \cdots = 1$

cc) $\cosh^2 x - \sinh^2 x = 1$ dd) $\int \tanh x \, dx = \operatorname{sech}^2 x$

ee) If $T_1(x)$ is the first degree Taylor polynomial for $f(x)$ at $x = a$, then the graphs of $f(x)$ and $T_1(x)$ have the same slope at $x = a$.

ff) If $\sqrt{5}$ is approximated by $T_1(5)$, where $T_1(x)$ is the Taylor polynomial of degree one for $\sqrt{x}$ at $a = 4$, then the approximation is larger than the exact value.

gg) $0.895 < e^{-0.1} < 0.905$ hh) $\sqrt{1+x^2} = 1 + x^2 + \cdots$ ii) $\int_0^{\pi/2} \sin^2 \theta d\theta = \int_0^{\pi/2} \cos^2 \theta d\theta$

jj) $e^{\pi i/4} + e^{-\pi i/4} = \sqrt{2}$ kk) $\cosh ix = \cos x$ ll) $\log(-1) = \pi i$ mm) $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^3 = 1$

nn) There are 56 ways of choosing 3 objects from a set of 8 objects, disregarding the order in which they are chosen.

oo) $\binom{6}{3} = 2$ pp) $\binom{10}{2} = \binom{10}{8}$ qq) $\binom{7}{3} + \binom{7}{4} = \binom{8}{4}$ rr) $\sum_{n=0}^{10} \binom{10}{n} = 1024$ ss) $\sum_{n=0}^{k} \binom{k}{n}(-1)^n = 0$

2. Evaluate the limit.

a) $1 + \frac{2024}{2025} + (\frac{2024}{2025})^2 + (\frac{2024}{2025})^3 + \cdots$ b) $\lim_{n\to\infty} \sum_{i=1}^{n} (1 + \frac{i}{n}) \cdot \frac{1}{n}$ c) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{1+\frac{i}{n}} \cdot \frac{1}{n}$

d) $\lim_{x\to 0} \frac{\sin x}{x}$ e) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ f) $\lim_{n\to\infty} (1 + \frac{x}{n})^{2n}$ g) $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$ h) $\lim_{h\to 0} \frac{(x+h)^4 - x^4}{h}$

i) $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ j) $\lim_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$ k) $\lim_{h\to 0} \frac{1}{h} \int_0^h f(x)\, dx$ l) $\lim_{h\to 0} \frac{1}{h^2} \int_0^h x f(x)\, dx$

---

integration

3. Find the antiderivative. Use power series in (c).

a) $\int e^{-x} dx$ b) $\int x e^{-x} dx$ c) $\int e^{-x^2} dx$ d) $\int x e^{-x^2} dx$ e) $\int x \sin x \, dx$ f) $\int e^{-x} \sin x \, dx$

g) $\int \frac{dx}{4x^2}$ h) $\int \frac{x}{4+x^2} dx$ i) $\int \frac{dx}{4+x^2}$ j) $\int \frac{dx}{\sqrt{4+x^2}}$ k) $\int \frac{dx}{4-x^2}$ l) $\int \frac{dx}{4x-x^2}$ m) $\int \frac{dx}{\sqrt{4x-x^2}}$

n) $\int \sin^2 x \, dx$ o) $\int \sin^3 x \, dx$ p) $\int \sin^4 x \, dx$

4. Evaluate the integral. a) $\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}} dx$ b) $\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$ c) $\int_{-\infty}^{\infty} (x-1)^2 \frac{1}{2\sqrt{2\pi}} e^{\frac{-(x-1)^2}{8}} dx$

5. Show that $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$. (hint: substitute $u = \frac{\pi}{2} - x$)

6. Determine whether the integral converges or diverges. If it converges, find the value.

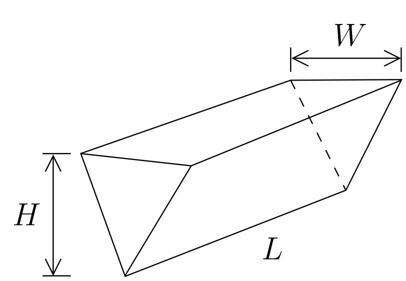
a) $\int_1^{\infty} \frac{dx}{x^2}$ b) $\int_1^{\infty} \frac{dx}{x}$ c) $\int_1^{\infty} \frac{dx}{x-1}$ d) $\int_0^1 \frac{dx}{x^2}$ e) $\int_0^1 \frac{dx}{\sqrt{x}}$ f) $\int_{-1}^1 \frac{dx}{x}$ g) $\int_0^{\infty} \frac{x^2 dx}{(1+x^2)^{7/2}}$

7. The electric potential due to a charged conducting sphere is $V(r) = \frac{q}{8\pi\epsilon_0 a} \int_{-a}^{a} \frac{dx}{(r^2-2rx+a^2)^{1/2}}$, where q is the total charge on the sphere, a is the radius of the sphere, and r is the distance from the center of the sphere to a point in space.

a) Evaluate $V(r)$. Consider two cases, $0 \le r \le a$ and $r > a$.

b) Sketch the graph of $V(r)$ for $r \ge 0$.

8. A tank with the indicated shape is full of water. The tank dimensions are height H m, length L m, and width W m, the water density is ρ kg/m^3, and the acceleration due to gravity is g m/s^2. Find the work done in pumping the water to the top of the tank.



9. Two identical ions repel each other with force $F = -\dfrac{q^2}{4\pi\epsilon_0 r^2}$, where q is the ion charge, r is the distance between the ions, ϵ_0 is the free-space permittivity and the negative sign indicates a repulsive force. (a) One ion is held fixed at $x = 0$; find the work done in moving another ion from

$x=3$ to $x=2$. (b) One ion is held fixed at $x=1$; find the work done in moving another ion from $x=3$ to $x=2$. (c) Two ions are held fixed at $x=0$ and $x=1$; find the work done in moving a third ion from $x=3$ to $x=2$. (d) A metal rod of uniform charge density is held fixed on the interval $0\le x\le 1$; the total charge on the rod is q; find the work done in moving an ion from $x=3$ to $x=2$.

10. a) Find the arclength of the graph of $f(x)=\frac{2}{3}x^{3/2}$ for $-1\le x\le 1$. b) Find the area of the surface obtained when the graph of $f(x)=\sin x$ for $0\le x\le\pi$ is rotated about the x-axis.

11. Sketch the region in the xy-plane and find the center of mass.

a) $\{(x,y): 0\le y\le x^2\ ,\ 0\le x\le 2\}$ b) $\{(x,y): x^2\le y\le 4\ ,\ 0\le x\le 2\}$

c) $\{(x,y): -1\le x\le 1,\ 0\le y\le 1-x^2\}$ d) $\{(x,y): 0\le y\le \frac{1}{1+x^2}\ ,\ 0\le x<\infty\}$

12. The lifetime of a light bulb is exponentially distributed with mean $\mu=1000$ hours. Find the probability that the lightbulb: a) fails in the first 200 hours, b) lasts more than 800 hours.

13. Let $f(x)=\begin{cases}\dfrac{1}{\pi\sqrt{x(1-x)}} & \text{if } 0<x<1,\\ 0 & \text{otherwise.}\end{cases}$ Sketch the graph and show that $f(x)$ is a valid pdf.

---

differential equations

14. Find the solution of the differential equation with initial condition $y(0)=y_0$. Sketch the solution for $t\ge 0$. Find $\lim_{t\to\infty} y(t)$.

a) $y'=-2y\ ,\ y_0=1$ b) $y'=1-2y\ ,\ y_0=0$ c) $y'=1-y^2\ ,\ y_0=0$ d) $y'=-ty\ ,\ y_0=1$

15. Consider the differential equation $y''=y$.

a) Show that $y(t)=c_1e^t+c_2e^{-t}$ is a solution, where c_1,c_2 are arbitrary constants.

b) Find the solution $y(t)$ subject to the initial conditions $y(0)=1$, $y'(0)=0$.

c) Repeat part (b) for initial conditions $y(0)=0$, $y'(0)=1$.

16. The cell count in a bacteria culture grows at a rate proportional to its size. After 30 minutes there are 200 cells and after 90 minutes there are 800 cells. (a) Find the initial cell count. (b) When will the cell count reach 6400? The answers should be expressed as integers.

17. Polonium-214 has a half-life of 1.4×10^{-4} s. If a sample has initial mass 40 mg, how long will it take for the mass to decay to 30 mg?

18. A tiger consumes 2500 calories per day and expends 20 calories per kg of its mass per day in daily activity. Assume 1 kg of the tiger's mass is equivalent to 10,000 calories. Formulate a differential equation for the mass of the tiger as a function of time, where $y(t)$ denotes the tiger's mass (kg) as a function of time t (day). Solve the equation subject to a general initial condition $y(0)=y_0$. What value does the tiger's mass approach as time increases? Sketch the graph of the tiger mass $y(t)$ for $t\ge 0$. Consider three cases, $y_0=100,125,150$, and sketch all three solutions on the same plot.

19. A thermometer at room temperature 70°F is placed in a patient's mouth. After one minute the thermometer reads 95°F and after two minutes it reads 100°F. Find the patient's temperature.

20. In a model for an epidemic, the rate of change of the infected population is proportional to the product of the number of people currently infected and the number of people not yet infected. In a town of 4000 inhabitants with a spreading epidemic, if 10 people are infected at the beginning of the week and 20 people are infected at the end of the week, how long does it take for half the population to be infected?

21. Consider the differential equation $y'=2y$ with initial condition $y_0=1$. We are interested in

the solution at time $t = 1$. Let u_n be the numerical solution given by Euler's method after n time steps with $\Delta t = \frac{1}{n}$. Find the expression for u_n and evaluate the limit $\lim_{\Delta t \to 0} u_n$.

---

series

22. Determine whether the series converges or diverges. Justify your answer.

a) $\sum_{n=1}^{\infty} \frac{1}{2n}$ b) $\sum_{n=1}^{\infty} \frac{1}{2^n}$ c) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ e) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

23. Express the repeating decimal as a rational number (i.e. a ratio of two integers).

a) $0.11111111\ldots$ b) $0.1212121212\ldots$ c) $0.4999999999\ldots$

24. Find the sum of the series. a) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ b) $\sum_{n=1}^{\infty} \frac{1}{3^n}$ c) $\sum_{n=1}^{\infty} \frac{n}{3^n}$

25. It is known that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Use this to evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

26. For each series, find a bound for $|s - s_{10}|$ using the estimates derived in class.

a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

27. Two students walk towards each other at 2 mi/hr starting from a separation of 20 miles. At the same time, a dog starts running back and forth between the students at 10 mi/hr. Let D be the total distance the dog has traveled when the students finally meet. Express D as an infinite series and find the sum of the series.

28. Winning a game of ping-pong requires a lead of two points, i.e. if the final score is tied, you must score two consecutive points in order to win the game. Suppose your probability of scoring a point is p, where $0 < p < 1$. If the final score is tied, find the probability you will eventually win the game. Evaluate for $p = \frac{1}{2}, \frac{1}{4}, \frac{3}{4}$. Interpret.

29. Start with the closed interval $[0, 1]$; remove the open interval $(\frac{1}{3}, \frac{2}{3})$; that leaves the two closed intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$; remove the middle third of those; that leaves four closed intervals; remove the middle third of those; continue the process indefinitely. The Cantor set is the set of points remaining after all the open intervals have been removed. (a) Sketch the remaining closed intervals for the first three steps. (b) Show that the total length of all the open intervals removed is 1. (c) Show that, nonetheless, the Cantor set contains infinitely many points.

---

power series, Taylor series

30. Find the interval of convergence of the power series; find the function $f(x)$ represented by the series; sketch the graph of $f(x)$ and indicate the interval of convergence on the x-axis.

a) $\sum_{n=0}^{\infty} x^n$ b) $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ c) $\sum_{n=0}^{\infty} (x-1)^n$ d) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ e) $\sum_{n=1}^{\infty} nx^n$

31. Find the power series representation for $f(x) = \frac{1}{1-x}$ about $x = \frac{1}{2}$.

32. Find the Taylor series for $\sinh x$ and $\cosh x$ about $x = 0$.

33. a) By squaring and adding the Taylor series for $\sin x$ and $\cos x$, find the Taylor series for $\sin^2 x + \cos^2 x$, up to the $O(x^6)$ term. b) Could you have predicted the answer to part (a)?

34. Find the Taylor series for $f(x) = e^{-x^2}$ about $x = 0$. Sketch $f(x)$, $T_0(x)$, $T_1(x)$, $T_2(x)$ in a neighborhood of $x = 0$. Label each curve.

35. Let $f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$ Evaluate the following limits.

a) $\lim_{x \to \infty} f(x)$ b) $\lim_{x \to 0^+} f(x)$ c) $\lim_{x \to 0^+} f'(x)$ d) $\lim_{x \to 0^+} f''(x)$ e) Sketch the graph of $f(x)$.

36. Find an approximate value for $\sqrt{10}$ which is accurate to within 0.005.

37. Use the Taylor series for $f(x) = \ln(1+x)$ about $x = 0$ to evaluate $\ln\frac{3}{2}$ to within 10^{-3}.

38. Find the first two nonzero terms in the Taylor series for $f(x)$ about $x = 0$.

a) $e^{-x}\sin x$ b) $(1-\cos x)/x$ c) $\tan x$ d) $\tan^{-1} x$

39. The Bernoulli numbers B_n are defined by $\dfrac{x}{e^x - 1} = \displaystyle\sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$. Find B_0, B_1, B_2.

40. Show that the following functions satisfy $f(0) = 0, f'(0) = 1$. Find $f''(0)$ in each case. If the functions are graphed in a neighborhood of $x = 0$, in what order do they appear (from top to bottom)? a) x b) $\sin x$ c) $\ln(1+x)$ d) $e^x - 1$

41. Recall the Bessel function of order zero, $J_0(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

a) Evaluate $\int_0^1 J_0(x)\,dx$ using 2 terms in the series. Find an upper bound for the error.

b) Show that $J_0(x)$ satisfies the differential equation $xy'' + y' + xy = 0$.

42. Let $f(t) = \sum_{n=0}^{\infty} t^n$.

a) Show that $f(t)$ satisfies the differential equation $y' = y^2$ with initial condition $y(0) = 1$.

b) Solve the differential equation for $f(t)$ by separation of variables. Do you recognize the result?

43. a) Show that $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,dx$ converges, but $\displaystyle\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx$ diverges.
(hint: sketch the graph, express the integral as a series, bound the terms in the series)

b) Find the value of $\int_0^{\infty} \frac{\sin x}{x}dx$. (hint: let $f(a) = \int_0^{\infty} \frac{\sin x}{x} e^{-ax}dx$ for $a \ge 0$, evaluate $f'(a)$, then find $f(a)$, and finally evaluate $f(0)$)

44. Use the 1st degree Taylor approximation for $\cos x$ about $x = 0$ to show that $|\cos\frac{\pi}{5} - 1| \le \frac{1}{2}(\frac{\pi}{5})^2$. Derive a more accurate result using the 3rd degree Taylor approximation.

45. Recall the error function, $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2} dt$. Find the first three terms in the Taylor series for $\operatorname{erf}(x)$ about $x = 0$.

46. In both cases below find the first three nonzero terms.

a) expand $\dfrac{a}{a+b}$ in powers of $\dfrac{a}{b}$ assuming $0 < a < b$

b) expand $\dfrac{\sqrt{R^2 - r^2}}{R}$ in powers of $\dfrac{r}{R}$ assuming $0 < r < R$

47. Show that $f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \cdots$ (an alternative form of the Taylor series).

48. The equation $\dfrac{x^2}{(1+\epsilon)^2} + y^2 = 1$ defines an ellipse in the xy-plane (assume $0 \le \epsilon < 1$).

a) Find the intercepts on the x-axis and y-axis. Sketch the ellipse.

b) Let $A(\epsilon)$ be the area of the ellipse. Express $A(\epsilon)$ as a definite integral.

c) Find the first 2 nonzero terms in the power series expansion of $A(\epsilon)$ about $\epsilon = 0$.

49. The gravitational potential induced by two point masses m_1, m_2 located at x_1, x_2 on the x-axis is $V(x) = -\dfrac{Gm_1}{|x - x_1|} - \dfrac{Gm_2}{|x - x_2|}$, where G is the gravitational constant. For $x \to \infty$, we have $V(x) \approx \dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{c}{x^3} + \cdots$, where $a, b, c, \ldots$ are constants that depend on m_1, m_2, x_1, x_2. Find a, b, c. (hint: set $y = 1/x$ and expand $V(x)$ in powers of y.)

50. The Lennard-Jones potential function, $V(r) = V_0\left(\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6}\right)$, describes the interaction between two particles (e.g. atoms, molecules), where V_0 and r_0 are positive constants, and $r \geq 0$ is the distance between the particles.
a) Find $\lim\limits_{r\to 0} V(r)$, $\lim\limits_{r\to\infty} V(r)$. b) Show that $V(r)$ has a minimum at $r = r_0$. c) Sketch the graph of $V(r)$ for $r \geq 0$. d) Find $T_2(r)$, the quadratic Taylor approximation for $V(r)$ at $r = r_0$.
e) Find the work done in separating two particles from $r = r_0$ to $r = \infty$ (this corresponds to dissociating a molecule). The force is $f(r) = -V'(r)$.

51. Use the 2nd degree Taylor approximation of $\sqrt{1+x^2}$ at $x = 0$ to approximate $\int_0^1 \sqrt{1+x^2}\,dx$. Find an upper bound for the error.

---

binomial series

52. Consider the expansion $\dfrac{1}{\sqrt{1-2ax+x^2}} = c_0 + c_1x + c_2x^2 + \cdots$, where a is a constant. Find c_0, c_1, c_2 in terms of a.

53. Recall the binomial expansion, $(a+b)^k = \sum\limits_{n=0}^{k} \binom{k}{n} a^{k-n}b^n$, where $k \geq 1$ is an integer.

a) Show that $\dbinom{k+1}{n+1} = \dbinom{k}{n} + \dbinom{k}{n+1}$.

b) Explain the connection between the formula in (a) and Pascal's triangle shown on the side.

c) Fill in the next two rows of the triangle and use this to expand $(a+b)^6$.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

---

complex numbers, polar coordinates

54. Express the complex number in Cartesian form $x + iy$ and polar form $re^{i\theta}$. Plot each number in the complex plane. a) $1+i$ b) $(1+i)^2$ c) $(1+i)^3$ d) $\frac{1}{1+i}$ e) $\sqrt{1+i}$

55. Compute $(1+i)^6$ two ways, (a) binomial expansion, (b) polar form.

56. Find the roots of the equation and plot them in the complex plane.
a) $z^2 + 2z - 2 = 0$ b) $z^2 + 2z + 2 = 0$ c) $z^2 = 1$ d) $z^3 = -1$ e) $z^4 = 1$ f) $e^z = 1$

57. a) Show that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
b) Use this to derive the double-angle formulas, $\sin 2\theta = 2\sin\theta\cos\theta, \cos 2\theta = \cos^2\theta - \sin^2\theta$.

58. Derive the addition formulas for sine and cosine, shown below. (hint : $e^{i(a+b)} = e^{ia}\cdot e^{ib}$)

$$\cos(a+b) = \cos a\,\cos b - \sin a\,\sin b\ ,\quad \sin(a+b) = \sin a\,\cos b + \cos a\,\sin b$$

59. a) Use integration by parts to find the antiderivative. $\int e^{ax}\cos bx\,dx$, $\int e^{ax}\sin bx\,dx$

b) Show that $e^{(a+ib)x} = e^{ax}\cos bx + ie^{ax}\sin bx$ and $\int e^{(a+ib)x}dx = \dfrac{a-ib}{a^2+b^2}e^{(a+ib)x}$.

c) Take the real and imaginary parts in (b) to rederive the formulas obtained in (a).

60. Using Euler's formula, $e^{ix} = \cos x + i\sin x$, derive a) $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$, b) $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$.
Using the results in (a) and (b), derive the following formulas.

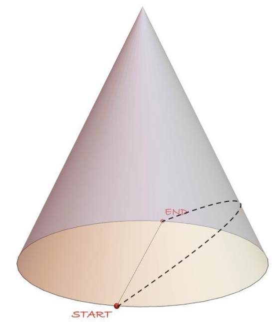
c) $\dfrac{d}{dx}\cos x = -\sin x$ d) $\dfrac{d}{dx}\sin x = \cos x$ e) $\cos^2 x + \sin^2 x = 1$ f) $\cos^2 x - \sin^2 x = \cos 2x$

g) $\sin 2x = 2\sin x\cos x$ h) $\cos^2 x = \dfrac{1}{2} + \dfrac{1}{2}\cos 2x$ i) $\cos^3 x = \dfrac{3}{4}\cos x + \dfrac{1}{4}\cos 3x$

---

bonus problem

61. A cone of height h and base radius 1 sits on the floor. An ant begins walking from a point on the base of the cone to the opposite point on the base along the shortest possible path; the Figure shows the ant's path as a dashed line. At some point on its journey the ant will reach a maximum altitude a from the base. We want to study how a depends on h.



a) If we cut the cone along a slant length from the tip of the cone to the ant's endpoint, the cone can be spread flat into a circular sector of radius R and angle θ. Find R and θ in terms of h, and sketch the ant's path through the sector. (hint: where does the ant's path begin on the sector?)

b) Show that the ant's maximum altitude in terms of h is $a(h) = h\left(1 - \cos\left(\frac{\pi}{2\sqrt{1+h^2}}\right)\right)$.

c) In the limit $h >> 1$, expand $a(h)$ to leading order in $1/h$, and use this to evaluate $\lim_{h\to\infty} a(h)$.