Math 156 Applied Honors Calculus II Final Exam Review Sheet Solutions Fall 2023

1. **True or False**

a) TRUE. $\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\cdots+\frac{n}{n}=\sum\limits_{i=1}^{n}\frac{i}{n}=\frac{1}{n}\cdot\sum\limits_{i=1}^{n}i=\frac{1}{n}\cdot\frac{n(n+1)}{2}=\frac{n+1}{2}$

b) TRUE. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}f'(x_i)\Delta x=\int_a^b f'(x)dx=f(b)-f(a)$ by FTC

c) FALSE. If n is doubled, then Δx decreases by a factor of $\frac{1}{2}$ and the error in the right-hand Riemann sum decreases by a factor of $\frac{1}{2}$, not $\frac{1}{4}$, as we saw in examples in class.

d) TRUE. Apply integration by parts.

$\int_0^1 f(x)g''(x)dx=f(x)g'(x)\Big|_0^1-\int_0^1 f'(x)g'(x)dx=-f'(x)g(x)\Big|_0^1+\int_0^1 f''(x)g(x)dx=\int_0^1 f''(x)g(x)dx$

The boundary terms vanish by the assumption $f(0)=f(1)=g(0)=g(1)=0$.

e) TRUE. $4x^2+9y^2=36\Rightarrow 9y^2=36-4x^2\Rightarrow y^2=4-\frac{4}{9}x^2\Rightarrow y=\pm2\sqrt{1-\frac{x^2}{9}},-3\le x\le 3$

$A=4\int_0^3 f(x)dx=4\int_0^3 2\sqrt{1-\frac{x^2}{9}}dx=8\int_0^3\sqrt{1-\frac{x^2}{9}}dx$

substitute $\sin\theta=\frac{x}{3},\cos\theta d\theta=\frac{dx}{3},\sqrt{1-\frac{x^2}{9}}=\cos\theta$

$A=8\int_0^{\pi/2}\cos\theta\cdot 3\cos\theta d\theta=24\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)d\theta=12(\theta+\frac{1}{2}\sin 2\theta)\Big|_0^{\pi/2}=12\cdot\frac{\pi}{2}=6\pi$

f) FALSE. $\int_0^\infty\frac{dx}{x^2}=\int_0^1\frac{dx}{x^2}+\int_1^\infty\frac{dx}{x^2}$, the integral from 0 to 1 diverges by the p-test, and even though the integral from 1 to ∞ converges by the p-test, the integral from 0 to ∞ diverges

g) FALSE. When the spring is stretched from length 20 cm to 30 cm, the work done is $\int_{20-L}^{30-L}kxdx=\int_{20-20}^{30-20}kxdx=\int_0^{10}kxdx=\frac{1}{2}kx^2\Big|_0^{10}=50k=2$ Joule. Then when the spring is stretched from length 30 cm to 40 cm, the work done is $\int_{30-L}^{40-L}kxdx=\int_{30-20}^{40-20}kxdx=\int_{10}^{20}kxdx=\frac{1}{2}kx^2\Big|_{10}^{20}=\frac{1}{2}k(400-100)=150k=3\cdot 50k=6$ Joule.

h) TRUE. $f(x)=\sqrt{x}=x^{1/2}\Rightarrow f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}\Rightarrow 1+(f'(x))^2=1+\frac{1}{4x}=\frac{4x+1}{4x}\Rightarrow S=\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}dx=\int_0^1 2\pi\sqrt{x}\cdot\frac{\sqrt{4x+1}}{2\sqrt{x}}dx=\pi\int_0^1(4x+1)^{1/2}dx=\pi\frac{2}{3}\cdot\frac{1}{4}(4x+1)^{3/2}\Big|_0^1=\frac{\pi}{6}(\sqrt{5}-1)$

i) FALSE.

<u>method 1</u>

$m=\int_a^b f(x)dx=\int_{-1}^1\cosh xdx=\sinh x|_{-1}^1=\sinh 1-\sinh(-1)=2\sinh 1$

$M_x=\frac{1}{2}\int_a^b f(x)^2dx=\frac{1}{2}\int_{-1}^1\cosh^2 xdx=\frac{1}{2}\int_{-1}^1(\frac{1}{2}+\frac{1}{2}\cosh 2x)dx=\frac{1}{2}\cdot(\frac{x}{2}+\frac{\sinh 2x}{4})|_{-1}^1$

$=\frac{1}{2}(\frac{1}{2}+\frac{\sinh 2}{4}-(\frac{-1}{2}+\frac{\sinh(-2)}{4}))=\frac{1}{2}(1+\frac{1}{2}\sinh 2)$

$M_y=\int_a^b xf(x)dx=\int_{-1}^1 x\cosh xdx$, set $u=x,dv=\cosh xdx\Rightarrow du=dx,v=\sinh x$

$=x\sinh x|_{-1}^1-\int_{-1}^1\sinh xdx=1\cdot\sinh 1-(-1)\cdot\sinh(-1)-\cosh x|_{-1}^1$

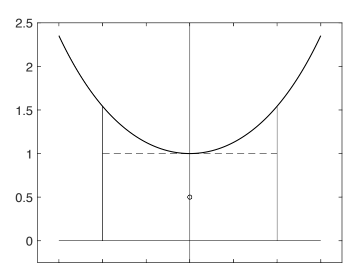
$=\sinh 1+\sinh(-1)-(\cosh 1-\cosh(-1))=\sinh 1-\sinh 1-(\cosh 1-\cosh 1)=0$

$\bar{x}=\frac{M_y}{m}=\frac{0}{2\sinh 1}=0:$ this part is true

$\bar{y}=\frac{M_x}{m}=\frac{\frac{1}{2}(1+\frac{1}{2}\sinh 2)}{2\sinh 1}=0.5985\ne\frac{1}{2}:$ can you prove this without a calculator?

<u>method 2</u>

The figure shows the graph of $y=\cosh x$. Since the region we are interested in is symmetric about the y-axis, it follows that the CM lies on the y-axis and hence $\bar{x}=0$. To show that $\bar{y}>\frac{1}{2}$, consider the rectangle $\{(x,y):-1\le x\le 1,0\le y\le 1\}$; the CM of the rectangle is $(0,\frac{1}{2})$ and the CM of the region we're interested in is clearly higher.



j) FALSE. A counterexample is an exponential distribution, $f(x)$ attains its maximum value at $x=0$ rather than $\mu=\frac{1}{c}$. (However the statement is true for a normal distribution.)

k) TRUE. $\int_{-\infty}^{\infty}(x-\mu)f(x)dx=\int_{-\infty}^{\infty}xf(x)dx-\int_{-\infty}^{\infty}\mu f(x)dx=\mu-\mu\cdot 1=0$

l) FALSE. After 100 years the sample has mass $\frac{1}{2}$kg, and after 400 years the sample has mass $\left(\frac{1}{2}\right)^4=\frac{1}{16}$ kg.

m) TRUE. continuous compounding $\Rightarrow y(t)=y_0e^{rt}=1000e^{0.05t}\Rightarrow y(2)=1000e^{0.05\cdot 2}=1000e^{0.1}$; hence we need to show that $1105<1000e^{0.1}<1112$
1st part : we need to show that $1105<1000e^{0.1}$; recall that $e^x=1+x+\frac{1}{2}x^2+\cdots$, so $e^{0.1}=1+0.1+\frac{1}{2}(0.1)^2+\cdots=1+0.1+0.005+\cdots$; then summing the first three terms yields $e^{0.1}=1.105+\cdots$, and since the remainder is positive, we have $e^{0.1}>1.105$, thus $1000e^{0.1}>1000\cdot 1.105=1105$
2nd part : we need to show that $1000e^{0.1}<1112$; this is equivalent to showing that $e^{0.1}<1.112$; in this case let us consider e^{-x}; it is fairly easy to see that $e^{-x}>1-x$ for $x\neq 0$; for example, consider the graph of e^{-x} and $1-x$; then let us set $x=0.1$, so that we have $e^{-0.1}>1-0.1$; this implies that $e^{-0.1}>0.9$; and this implies that $e^{0.1}<\frac{1}{0.9}=\frac{1}{\frac{9}{10}}=\frac{10}{9}=1.1111\cdots<1.112$; so we're done

n) FALSE. $y(t)=0$ is a constant solution of the differential equation, but it is unstable by looking at the phase plane.

o) FALSE. This is only true if the constant solution is stable.

p) FALSE. If the step size Δt decreases, then the error also decreases.

q) FALSE. A counterexample is $a_n=\frac{1}{n}$, $b_n=n^2$.

r) FALSE. A counterexample is $a_n=\frac{1}{n^2}, b_n=\frac{1}{n}$, so that $\sum\limits_{n=0}^{\infty}a_n$ converges, but $\sum\limits_{n=0}^{\infty}b_n$ diverges.

s) FALSE.
Method 1: draw a graph of $y=\frac{1}{x^2}$ for $x\geq 1$ and note that $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ is a left-hand Riemann sum for $\int_1^{\infty}\frac{dx}{x^2}$, and hence the sum is larger than the integral.
Method 2: $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}, \int_1^{\infty}\frac{dx}{x^2}=\frac{-1}{x}\Big|_1^{\infty}=1$

t) FALSE.
Method 1: The error bound for a convergent alternating series says that $|s-s_n|<a_{n+1}$. If we set $n=1$, then $|s-s_1|<a_2\Rightarrow |s-1|<\frac{1}{2}\Rightarrow\frac{1}{2}<s<\frac{3}{2}$, so $s\neq 0$.
Method 2: In class we showed that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\ln 2\neq 0$.

u) FALSE.
$\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum\limits_{n=1}^{\infty}(\frac{1}{n}-\frac{1}{n+1})=\sum\limits_{n=1}^{\infty}\frac{1}{n}-\sum\limits_{n=1}^{\infty}\frac{1}{n+1}=\infty-\infty=0$
The 1st step is correct, but the 2nd step is incorrect; the original sum is equal to 1 (as shown on homework), so it is incorrect to write $1=\infty-\infty$.

v) FALSE. The AST cannot be used to show that a series diverges.

w) FALSE. $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\frac{n^2}{(n+1)^2}=1\Rightarrow$ the ratio test is inconclusive

x) FALSE. A counterexample is $c_n=\frac{(-1)^{n+1}}{n+1}$. In that case $\sum\limits_{n=0}^{\infty}c_nx^n=\sum\limits_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}x^n$, so $x=1$ yields the alternating harmonic series which converges by the AST, and $x=-1$ yields the harmonic series which diverges as shown in class.

y1) TRUE. The radius of convergence is at least $R=1$, so the interval of convergence is at least $0<x\leq 2$, which contains $x=\frac{1}{2}$.

y2) FALSE. geometric series, $r=2(x-1)$, converges for $-1<r<1\Leftrightarrow -1<2(x-1)<1\Leftrightarrow -\frac{1}{2}<x-1<\frac{1}{2}\Leftrightarrow\frac{1}{2}<x<\frac{3}{2}$, diverges at end points $x=\frac{1}{2},\frac{3}{2}$

z1) TRUE. Start from the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$.

Replace x by $-x$ to get $\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1-(-x)} = \frac{1}{1+x}$ for $|x| < 1$.

Differentiate to get $\sum_{n=1}^{\infty}(-1)^n n x^{n-1} = -\frac{1}{(1+x)^2} \Rightarrow \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} = \frac{1}{(1+x)^2}$ for $|x| < 1$.

Substitute $j = n-1$ to get $\sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} = \sum_{j=0}^{\infty}(-1)^{j+2}(j+1)x^j = \frac{1}{(1+x)^2}$ for $|x| < 1$.

Use the fact that $(-1)^{j+2} = (-1)^j$ to get $\sum_{j=0}^{\infty}(-1)^j (j+1)x^j = \frac{1}{(1+x)^2}$ for $|x| < 1$.

Finally replace j by n to get $\sum_{n=0}^{\infty}(-1)^n (n+1)x^n = \frac{1}{(1+x)^2}$ for $|x| < 1$. Now check the endpoints.

$x = 1 \Rightarrow \sum_{n=0}^{\infty}(-1)^n(n+1)$ diverges because $\lim_{n\to\infty} a_n \neq 0$.

$x = -1 \Rightarrow \sum_{n=0}^{\infty}(-1)^n(n+1)(-1)^n$ also diverges because $\lim_{n\to\infty} a_n \neq 0$. So the ioc is $-1 < x < 1$.

Another method is to show that the Taylor coefficients of $f(x) = \frac{1}{(1+x)^2}$ are $\frac{f^{(n)}(0)}{n!} = (-1)^n(n+1)$.

z2) FALSE. Don't try to find $f^{(3)}(0), f^{(6)}(0)$ directly. Instead use the Taylor series to derive them, since the general form of a Taylor series is $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_n = \frac{f^{(n)}(0)}{n!} \Rightarrow f^{(n)}(0) = n! \cdot c_n$. Since $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$, then $e^{-x^2} = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots$. Then $c_3 = 0$, since there is no x^3 term, and hence $f^{(3)}(0) = 0$. Then $c_6 = -\frac{1}{6}$, thus $f^{(6)}(0) = 6! \cdot c_6 = 720 \cdot \left(-\frac{1}{6}\right) = -120$. Thus the statement is false.

z3) TRUE.
part 1 : Since $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$, then setting $x = 1$, we have $e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots$, and it is clear that $e > 2$.
part 2 : Setting $x = -1$ in the power series for e^x, we have $e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{3!} + \cdots$, then using the error bound for a convergent alternating series we have $|e^{-1} - \frac{1}{2}| < \frac{1}{6} \Rightarrow \frac{1}{3} < e^{-1} < \frac{2}{3}$, then $e^{-1} > \frac{1}{3} \Rightarrow e < 3$.

aa) TRUE. $e^x = 1 + x + \frac{1}{2}x^2 + \cdots \Rightarrow e^{-x^2} = 1 + (-x^2) + \frac{1}{2}(-x^2)^2 + \cdots = 1 - x^2 + \frac{1}{2}x^4 - \cdots$
$\Rightarrow \int_0^1 e^{-x^2}dx = \int_0^1 \left(1 - x^2 + \frac{1}{2}x^4 - \cdots\right)dx = \left(x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \cdots\right)\Big|_0^1 = 1 - \frac{1}{3} + \frac{1}{10} + \cdots$
This is an alternating series; it can be seen that the three conditions of the alternating series test are satisfied; so the series converges and we can apply the error bound for alternating series.
error bound : $|s - s_n| < a_{n+1}$, let us consider the 1st partial sum
$\Rightarrow |\int_0^1 e^{-x^2}dx - 1| < \frac{1}{3} \Rightarrow -\frac{1}{3} < \int_0^1 e^{-x^2}dx - 1 < \frac{1}{3} \Rightarrow \frac{2}{3} < \int_0^1 e^{-x^2}dx < \frac{4}{3}$

bb) TRUE. Since the series is $\sin\frac{\pi}{2}$ (note that $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$) and $\sin\frac{\pi}{2} = 1$.

cc) TRUE. $\cosh^2 x - \sinh^2 x = \left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2 = \frac{e^{2x}+2+e^{-2x}}{4} - \frac{e^{2x}-2+e^{-2x}}{4} = 1$.

dd) FALSE. $\int \tanh x dx = \int \frac{\sinh x}{\cosh x}dx = \int \frac{1}{\cosh x}d\cosh x = \ln(\cosh x) \neq \mathrm{sech}^2 h$. Note that $(\tanh x)' = \mathrm{sech}^2 x$.

ee) TRUE. Note that $T_1(x) = f(a) + f'(a)(x-a)$, thus $T_1'(a) = f'(a)$.

ff) TRUE. With $f(x) = \sqrt{x}, a = 4$, we have $f'(x) = \frac{1}{2\sqrt{x}}$, and $f(a) = \sqrt{4} = 2, f'(a) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$, so $T_1(x) = f(a) + f'(a)(x-a) = 2 + \frac{1}{4}(x-4) = 1 + \frac{1}{4}x$, then sketch the graph of $\sqrt{x}$ and $T_1(x)$, and since $\sqrt{x}$ is concave down, we see that $T_1(5) > \sqrt{5}$.

gg) TRUE. $e^x = 1 + x + \frac{1}{2}x^2 + \cdots$, so $e^{-0.1} = 1 + (-0.1) + \frac{1}{2}(-0.1)^2 + \cdots = 1 - 0.1 + 0.005 - \cdots$, so $|e^{-0.1} - 0.9| \leq 0.005$, so $0.895 \leq e^{-0.1} \leq 0.905$.

hh) FALSE. Using the binomial series $(1+x)^k = 1 + kx + \cdots$, replace x with x^2, then $(1+x^2)^k = 1 + kx^2 + \cdots$, and then setting $k = \frac{1}{2}$, we have $\sqrt{1+x^2} = (1+x^2)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \cdots$.

ii) TRUE.
part 1 : $\int_0^{\pi/2} \sin^2\theta d\theta = \int_0^{\pi/2}(\frac{1}{2} - \frac{1}{2}\cos 2\theta)d\theta = (\frac{1}{2}\theta - \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta)\Big|_0^{\pi/2} = \frac{\pi}{4}$
part 2 : $\int_0^{\pi/2} \cos^2\theta d\theta = \int_0^{\pi/2}(\frac{1}{2} + \frac{1}{2}\cos 2\theta)d\theta = (\frac{1}{2}\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta)\Big|_0^{\pi/2} = \frac{\pi}{4}$
This can also be shown using integration by parts.
jj) TRUE. $e^{\pi i/4} + e^{-\pi i/4} = (\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) + (\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}) = \big(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) + (\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}) = \sqrt{2}$
kk) TRUE. $\cosh ix = \frac{e^{ix}+e^{-ix}}{2} = \frac{\cos x + i\,\sin x + \cos x - i\,\sin x}{2} = \cos x$
ll) TRUE. $e^{\pi i} = \cos\pi + i\,\sin\pi = -1 \Rightarrow \log(e^{\pi i}) = \log(-1) \Rightarrow \pi i\log(e) = \log(-1) \Rightarrow \pi i = \log(-1)$
Note that this problem uses the complex logarithm defined by $\log z = \log re^{i\theta} = \ln r + i\theta$.
mm) TRUE.
method 1: $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^3 = (-\frac{1}{2} + i\frac{\sqrt{3}}{2})\cdot(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^2 = (-\frac{1}{2} + i\frac{\sqrt{3}}{2})\cdot(\frac{1}{4} - i\frac{\sqrt{3}}{2} - \frac{3}{4})$
$= (-\frac{1}{2} + i\frac{\sqrt{3}}{2})\cdot(-\frac{1}{2} - i\frac{\sqrt{3}}{2}) = \frac{1}{4} + i\frac{\sqrt{3}}{4} - i\frac{\sqrt{3}}{4} + \frac{3}{4} = 1$
method 2: $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^3 = (\cos 2\pi/3 + i\sin 2\pi/3)^3 = (e^{2\pi i/3})^3 = e^{6\pi i/3} = e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$
nn) TRUE. $\binom{8}{3} = \frac{8!}{3!5!} = \frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1} = 8\cdot 7 = 56$
oo) FALSE. $\binom{6}{3} = \frac{6!}{3!3!} = \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1} = 20$
pp) TRUE. $\binom{10}{2} = \frac{10!}{2!\cdot 8!} = \frac{10!}{8!\cdot 2!} = \binom{10}{8}$
qq) TRUE. $\binom{7}{3} + \binom{7}{4} = \frac{7!}{3!\cdot 4!} + \frac{7!}{4!\cdot 3!} = \frac{2\cdot 7!}{3!\cdot 4!} = \frac{8\cdot 7!}{4\cdot 3!\cdot 4!} = \frac{8!}{4!\cdot 4!} = \binom{8}{4}$
rr) TRUE. $(a+b)^k = \sum_{n=0}^{k}\binom{k}{n}a^{k-n}b^n$, set $a = b = 1, k = 10 \Rightarrow \sum_{n=0}^{10}\binom{10}{n} = (1+1)^{10} = 2^{10} = 1024$
ss) TRUE. $(a+b)^k = \sum_{n=0}^{k}\binom{k}{n}a^{k-n}b^n$, set $a = 1, b = -1 \Rightarrow \sum_{n=0}^{k}\binom{k}{n}(-1)^n = (1+(-1))^k = 0^k = 0$

Question 2 Solution
a) geometric series, $r = \frac{2022}{2023}$: sum $= \frac{1}{1-r} = \frac{1}{1-\frac{2022}{2023}} = \frac{2023}{2023-2022} = 2023$
b) $\lim_{n\to\infty}\sum_{i=1}^{n}(1+\frac{i}{n})\cdot\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx = \int_0^1(1+x)dx = \Big(x + \frac{x^2}{2}\Big)\Big|_0^1 = \frac{3}{2}$
$a = 0, b = 1, \Delta x = \frac{b-a}{n} = \frac{1}{n}, x_i = a + i\Delta x = \frac{i}{n}, f(x_i) = 1 + \frac{i}{n}, f(x) = 1 + x$
c) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\frac{i}{n}}\cdot\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx = \int_0^1\frac{1}{1+x}dx = \ln(1+x)|_0^1 = \ln 2 - \ln 1 = \ln 2$
$a = 0, b = 1, \Delta x = \frac{b-a}{n} = \frac{1}{n}, x_i = a + i\Delta x = \frac{i}{n}, f(x_i) = \frac{1}{1+\frac{i}{n}}, f(x) = \frac{1}{1+x}$
d) By L'Hospital's rule, $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{(\sin x)'}{(x)'} = \lim_{x\to 0}\frac{\cos x}{1} = \cos 0 = 1.$
e) By L'Hospital's rule, $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{(1-\cos x)'}{(x^2)'} = \lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}.$
f) Note that $\lim_{n\to\infty}\big(1+\frac{x}{n}\big)^{2n} = (\lim_{n\to\infty}\big(1+\frac{x}{n}\big)^n)^2 = (e^x)^2 = e^{2x}$
g) By L'Hospital's rule $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0}\frac{(\sqrt{1+x}-1)'}{x'} = \lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$
h) $\lim_{h\to 0}\frac{(x+h)^4 - x^4}{h} = \lim_{h\to 0}\frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} = \lim_{h\to 0}\frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \lim_{h\to 0}(4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3$ (one can use L'hospital rule, note that 'h' is variable here, regard x as constant)
i) By L'Hospital's rule, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f'(x+h)}{1} = f'(x)$.
j) Using L'Hospital's rule twice, $\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} = \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} = \lim_{t\to 0}\frac{f''(x+h)+f''(x-h)}{2} = f''(x)$.
k) $\lim_{h\to 0}\frac{\int_0^h f(x)dx}{h} \xrightarrow{\text{L'Hospital Rule}} \lim_{h\to 0}\frac{(\int_0^h f(x)dx)'}{(h)'} = \lim_{h\to 0}\frac{f(h)}{1} = f(0)$
l) $\lim_{h\to 0}\frac{\int_0^h xf(x)dx}{h^2} \xrightarrow{\text{L'Hospital Rule}} \lim_{h\to 0}\frac{(\int_0^h xf(x)dx)'}{(h^2)'} = \lim_{h\to 0}\frac{hf(h)}{2h} = \lim_{h\to 0}\frac{f(h)}{2} = \frac{f(0)}{2}$

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integration

Question 3 Solution

a) $\int e^{-x}dx = -e^{-x}$

b) integration by parts : $\int xe^{-x}dx = x \cdot -e^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} = -(1+x)e^{-x}$

c) $e^x = \sum\limits_{n=0}^{\infty} \frac{1}{n!}x^n \Rightarrow e^{-x^2} = \sum\limits_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n = \sum\limits_{n=0}^{\infty} \frac{1}{n!}(-1)^n x^{2n}$

$\Rightarrow \int e^{-x^2}dx = \int \sum\limits_{n=0}^{\infty} \frac{1}{n!}(-1)^n x^{2n}dx = \sum\limits_{n=0}^{\infty} \frac{1}{n!}(-1)^n \frac{1}{2n+1}x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \cdots.$

d) substitute $u = x^2, du = 2xdx$: $\int xe^{-x^2}dx = \int e^{-u} \cdot \frac{1}{2}du = -\frac{1}{2}e^{-u} = -\frac{1}{2}e^{-x^2}$

e) integration by parts : $\int x \sin x dx = \int x(-1)d\cos x = -\int x d\cos x = -x\cos x + \int \cos x dx = -x\cos x + \sin x = \sin x - x\cos x$

f) use integration by parts twice

once : $\int e^{-x}\sin x dx = \int e^{-x}(-1)d\cos x = -e^{-x}\cos x + \int \cos x de^{-x} = -e^{-x}\cos x - \int e^{-x}\cos x dx$

twice : $\int e^{-x}\cos x dx = \int e^{-x}d\sin x = e^{-x}\sin x - \int \sin x de^{-x} = e^{-x}\sin x + \int e^{-x}\sin x dx$

$\Rightarrow \int e^{-x}\sin x dx = -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin x dx \quad \Rightarrow \quad 2\int e^{-x}\sin x = -e^{-x}\cos x - e^{-x}\sin x$

$\Rightarrow \int e^{-x}\sin x = \frac{1}{2}\left(-e^{-x}\cos x - e^{-x}\sin x\right) = -\frac{1}{2}e^{-x}(\cos x + \sin x)$

g) $\int \frac{dx}{4x^2} = -\frac{1}{4x}$

h) substitute $u = 4 + x^2, du = 2xdx$, $\int \frac{x}{4+x^2}dx = \int \frac{1}{u} \cdot \frac{du}{2} = \int \frac{du}{2u} = \frac{1}{2}\ln u = \frac{1}{2}\ln(4+x^2)$

i) trig substitution, $x = 2\tan\theta$, $dx = 2\sec^2\theta d\theta$

$\int \frac{dx}{4+x^2} = \int \frac{2\sec^2\theta}{4+4\tan^2\theta}d\theta = \frac{1}{2}\int \frac{\sec^2\theta}{1+\tan^2\theta}d\theta = \frac{1}{2}\int \frac{\sec^2\theta}{1+\frac{\sin^2\theta}{\cos^2\theta}}d\theta = \frac{1}{2}\int \frac{\sec^2\theta\cos^2\theta}{\cos^2\theta+\sin^2\theta}d\theta = \frac{1}{2}\int d\theta = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}\frac{x}{2}$

j) substitute $x = 2\sinh\theta, dx = 2\cosh\theta d\theta$ and use $1 + \sinh^2\theta = \cosh^2\theta$

$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\cosh\theta}{\sqrt{4+4\sinh^2\theta}}d\theta = \int \frac{2\cosh\theta}{\sqrt{4\cosh^2\theta}}d\theta = \int \frac{2\cosh\theta}{2\cosh\theta}d\theta = \int d\theta = \theta = \mathrm{arcsinh}\frac{x}{2} = \sinh^{-1}\frac{x}{2}$

k) Using partial fractions

$\int \frac{dx}{4-x^2} = \int \frac{dx}{(2+x)(2-x)} = \int \left(\frac{1/4}{2+x} + \frac{1/4}{2-x}\right)dx = \int \frac{1/4}{2+x}dx + \int \frac{1/4}{2-x}dx = \frac{1}{4}\ln|2+x| - \frac{1}{4}\ln|2-x|$

l) partial fractions again

$\int \frac{dx}{4x-x^2} = \int \frac{dx}{x(4-x)} = \int \left(\frac{1/4}{x} + \frac{1/4}{4-x}\right)dx = \int \frac{1/4}{x}dx + \int \frac{1/4}{4-x}dx = \frac{1}{4}\ln|x| - \frac{1}{4}\ln|4-x|$

m) To compute $\int \frac{dx}{\sqrt{4x-x^2}}$, first complete the square as follows,

$4x - x^2 = -x^2 + 4x = -(x^2 - 4x) = -(x^2 - 4x + 4 - 4) = -((x-2)^2 - 4) = 4 - (x-2)^2$,

then apply the substitution $u = x - 2, du = dx$,

$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{du}{\sqrt{4-(x-2)^2}} = \int \frac{du}{\sqrt{4-u^2}}$,

and then apply the trig substitution, $\sin\theta = u/2, \cos\theta d\theta = du/2, \cos\theta = \sqrt{4-u^2}/2$,

$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{du}{\sqrt{4-u^2}} = \int \frac{2\cos\theta d\theta}{2\cos\theta} = \int d\theta = \theta = \sin^{-1}(u/2) = \sin^{-1}\frac{x-2}{2}$.

n) $\int \sin^2 x dx$

method 1 : integrate by parts and use the equalities $\sin^2 x + \cos^2 x = 1, \sin 2x = 2\cos x \sin x$

$\int \sin^2 x dx = \int \sin x \cdot \sin x dx = \int \sin x \, d(-\cos x) = -\sin x \cdot \cos x - \int(-\cos x)d\sin x$

$= -\sin x \cdot \cos x + \int \cos^2 x dx = -\sin x \cdot \cos x + \int(1 - \sin^2 x)dx = -\sin x \cdot \cos x + \int dx - \int \sin^2 x dx$

$\Rightarrow 2\int \sin^2 x dx = -\sin x \cdot \cos x + x \Rightarrow \int \sin^2 x dx = -\frac{1}{2}\sin x\cos x + \frac{x}{2} = \frac{x}{2} - \frac{1}{4}\sin 2x$

method 2 : using $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \Rightarrow \int \sin^2 x dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx = \frac{x}{2} - \frac{1}{4}\sin 2x$

o) $\int \sin^3 x dx = \int \sin^2 x \cdot \sin x dx = \int(1 - \cos^2 x)\sin x dx = (\text{set } u = \cos x, du = -\sin x dx) = \int(1-u^2) \cdot -du = \int(-1+u^2)du = -u + \frac{1}{3}u^3 = -\cos x + \frac{1}{3}\cos^3 x$

p) using integration by parts will be too complicated, using $\sin^2 x = \frac{1-\cos 2x}{2}$, $\sin 2x = 2\cos x \sin x$, $(\cos 2x)^2 = (1 + \cos 4x)/2$

Thus $\int \sin^4 x = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \int \frac{1-2\cos 2x+\cos^2 2x}{4}dx = \int \frac{1-2\cos 2x+1-\sin^2 2x}{4}dx$

$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x - \frac{1}{4}\sin^2 2x\right)dx = \frac{x}{2} - \frac{1}{4}\sin 2x - \frac{1}{8}\int \sin^2 2x d(2x)$

$\xrightarrow{using\ 3\ i) x\to 2x} \frac{x}{2}-\frac{1}{4}\sin 2x-\frac{1}{8}\left(\frac{2x}{2}-\frac{\sin 4x}{4}\right)=\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x$

Question 4 Solution

a) $\int_0^{2\sqrt{3}}\frac{x^3}{\sqrt{16-x^2}}\,dx$: use trig substitution $\sin\theta=x/4, \cos\theta d\theta=dx/4, \cos\theta=\sqrt{16-x^2}/4$

$\int_0^{2\sqrt{3}}\frac{x^3}{\sqrt{16-x^2}}\,dx=\int_0^{\pi/3}\frac{64\sin^3\theta}{4\cos\theta}4\cos\theta d\theta=64\int_0^{\pi/3}\sin^3\theta d\theta=64(-\cos\theta+\frac{1}{3}\cos^3\theta)\Big|_0^{\pi/3}$: problem 3(o)

$=64\left(-\frac{1}{2}+\frac{1}{3}(\frac{1}{2})^3-(-1+\frac{1}{3})\right)=64\left(-\frac{1}{2}+\frac{1}{24}+1-\frac{1}{3}\right)=64\left(\frac{-12+1+24-8}{24}\right)=64\cdot\frac{5}{24}=\frac{40}{3}$

b) $\int_{-\infty}^{\infty}x^2\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx$

method 1 : recall that $\int_{-\infty}^{\infty}(x-\mu)^2\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}}dx=\sigma^2$, this was stated on page 37 of the notes and was proven on hw7; it relates to the normal pdf, $f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, with mean μ and standard deviation σ; hence if we set $\mu=0,\sigma=1$, then we get $\int_{-\infty}^{\infty}x^2\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx=\sigma^2=1$

method 2 : integrate by parts, $u=x, dv=xe^{-\frac{x^2}{2}}dx\Rightarrow du=dx, v=-e^{-\frac{x^2}{2}}$

$\int_{-\infty}^{\infty}x^2\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx=\frac{1}{\sqrt{2\pi}}\cdot\int_{-\infty}^{\infty}x^2e^{\frac{-x^2}{2}}dx=\frac{1}{\sqrt{2\pi}}\left(x\cdot -e^{-\frac{x^2}{2}}\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}dx\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}dx$

substitute $u=\frac{x}{\sqrt{2}}\Rightarrow du=\frac{dx}{\sqrt{2}}$, then $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-u^2}\sqrt{2}du=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-u^2}du=1$, where we have used the fact that $\int_{-\infty}^{\infty}e^{-x^2}dx=\sqrt{\pi}$

c) $\int_{-\infty}^{\infty}(x-1)^2\frac{1}{2\sqrt{2\pi}}e^{\frac{-(x-1)^2}{8}}dx$, similar to 4b with $\mu=1,\sigma=2$, so the integral is $\sigma^2=4$

Question 5 Solution

Consider the given integral $\int_0^{\pi/2}\frac{\sin x}{\cos x+\sin x}dx$ and substitute $u=\frac{\pi}{2}-x$, $du=-dx$.

$\int_0^{\pi/2}\frac{\sin x}{\cos x+\sin x}du=\int_{\pi/2}^{0}\frac{\sin(\frac{\pi}{2}-u)}{\sin(\frac{\pi}{2}-u)+\cos(\frac{\pi}{2}-u)}(-1)du=-\int_{\pi/2}^{0}\frac{\cos u}{\cos u+\sin u}du=\int_0^{\pi/2}\frac{\cos u}{\cos u+\sin u}du$

$=\int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}dx$

then $\int_0^{\pi/2}\frac{\sin x}{\cos x+\sin x}dx+\int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}dx=\int_0^{\pi/2}\frac{\sin x+\cos x}{\cos x+\sin x}dx=\int_0^{\pi/2}1\cdot dx=\frac{\pi}{2}$

and finally $\int_0^{\pi/2}\frac{\sin x}{\cos x+\sin x}dx=\int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}dx=\frac{\pi}{4}$

Question 6 Solution

a) convergent by $p-$test, $\int_1^{\infty}\frac{1}{x^2}dx=\int_0^{\infty}\left(-\frac{1}{x}\right)'dx=-\frac{1}{x}\Big|_1^{\infty}=1$

b) divergent by $p-$test, $\int_1^{\infty}\frac{dx}{x}=\ln x|_1^{\infty}=\infty$

c) $\int_1^{\infty}\frac{dx}{x-1}=\int_0^{\infty}\frac{dy}{y}$ divergent by $p-$test

d) divergent by p-test, $\int_0^1\frac{dx}{x^2}=-\frac{1}{x}\Big|_0^1=\infty$

e) convergent by p-test, $\int_0^1\frac{dx}{\sqrt{x}}=2\sqrt{x}\Big|_0^1=2$

f) divergent by p-test, since $\int_{-1}^{0}\frac{dx}{x}$ and $\int_0^1\frac{dx}{x}$ both diverge

g) $\int_0^{\infty}\frac{x^2dx}{(1+x^2)^{7/2}}$ converges

Note that $\frac{x^2}{(1+x^2)^{7/2}}=0$ at $x=0$, which implies that the integral is improper only because of the upper limit ∞, and since $x^2/(1+x^2)^{7/2}\sim 1/x^5$ for $x\to\infty$, the integral converges.

To make this rigorous, proceed as follows.

$\int_0^{\infty}\frac{x^2dx}{(1+x^2)^{7/2}}=\int_0^1\frac{x^2dx}{(1+x^2)^{7/2}}+\int_1^{\infty}\frac{x^2dx}{(1+x^2)^{7/2}}$

The 1st integral is a proper integral, so it converges. The 2nd integral is an improper integral and we can apply the comparison test, $\int_1^{\infty}\frac{x^2dx}{(1+x^2)^{7/2}}\le\int_1^{\infty}\frac{x^2dx}{(x^2)^{7/2}}=\int_1^{\infty}\frac{dx}{x^5}$, which converges by the p-test, hence the original integral converges.

To compute the integral use trig substitution, $\tan\theta=x, \sec^2\theta d\theta=dx, \sin\theta=\frac{x}{\sqrt{1+x^2}}, \cos\theta=\frac{1}{\sqrt{1+x^2}}$.

$\int_0^{\infty}\frac{x^2dx}{(1+x^2)^{7/2}}=\int_0^{\infty}\frac{x^2}{1+x^2}\cdot\frac{1}{(1+x^2)^{5/2}}dx=\int_0^{\pi/2}\sin^2\theta\cos^5\theta\sec^2\theta d\theta=\int_0^{\pi/2}\sin^2\theta\cos^3\theta d\theta$

$=\int_0^{\pi/2}\sin^2\theta(1-\sin^2\theta)\cos\theta d\theta=(u=\sin\theta, du=\cos\theta d\theta)$

$= \int_0^1 u^2(1-u^2)du = \left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right)\Big|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{5-3}{15} = \frac{2}{15}$

Question 7 Solution

substitute $u = r^2 - 2rx + a^2$, so $du = -2rdx$

limits $x = -a \Rightarrow u = r^2 - 2r(-a) + a^2 = r^2 + 2ra + a^2 = (r+a)^2$, $x = a \Rightarrow u = r^2 - 2ra + a^2 = (r-a)^2$

$V(r) = \frac{q}{8\pi\epsilon_0 a}\int_{-a}^{a}\frac{dx}{\sqrt{r^2-2rx+a^2}} = \frac{q}{8\pi\epsilon_0 a}\int_{(r+a)^2}^{(r-a)^2}\frac{-\frac{1}{2r}du}{\sqrt{u}} = -\frac{q}{16\pi\epsilon_0 ar}\cdot 2\sqrt{u}\Big|_{(r+a)^2}^{(r-a)^2} = -\frac{q}{8\pi\epsilon_0 ar}\left(|r-a| - |r+a|\right)$

$= \begin{cases} \frac{q}{4\pi\epsilon_0 a} & \text{if } 0 \le r \le a \\ \frac{q}{4\pi\epsilon_0 r} & \text{if } r > a \end{cases}$

Question 8 Solution

Let x be the vertical coordinate with $x = 0$ at the bottom of the tank (so positive x is up); divide the water into layers of width $\Delta x = \frac{H}{n}$; the ith layer is at height $x_i = i\Delta x$; each layer is a rectangular box with volume $w_i L\Delta x$, where w_i is the width of the ith layer; the force on the ith layer is mass $\cdot$ acceleration $= \rho w_i L\Delta x \cdot g$; the ith layer is raised a distance $H - x_i$; by similar triangles we find that $H/W = x_i/w_i$, so $w_i = x_i W/H$; the work done is

$\lim_{n\to\infty}\sum_{i=1}^{n}\rho w_i L\Delta x\cdot g\cdot(H - x_i) = \int_0^H \rho gLx\frac{W}{H}(H-x)dx = \rho gL\frac{W}{H}\int_0^H(Hx - x^2)dx$

$= \rho gL\frac{W}{H}\left(\frac{1}{2}Hx^2 - \frac{1}{3}x^3\right)\Big|_0^H = \rho gL\frac{W}{H}\left(\frac{1}{2}H^3 - \frac{1}{3}H^3\right) = \frac{1}{6}\rho gLWH^2$

alternative method : The entire mass of water in the tank is $\rho\cdot\frac{1}{2}WHL$. Imagine that the entire water mass is concentrated at the CM of the tank. The force acting on the mass then is $\frac{1}{2}\rho gWHL$. The CM of the tank lies at a distance $\frac{1}{3}H$ from the top of the tank. Then the work done in raising the mass to the top of the tank is mass $\cdot$ distance $= \frac{1}{2}\rho gWHL\cdot\frac{1}{3}H = \frac{1}{6}\rho gLWH^2$.

Question 9 Solution

a) On the x-axis, one ion is held fixed at $x = 0$, so the distance between the ions is x, so replace r in $F = -\frac{q^2}{4\pi\epsilon_0 r^2}$ by x.

Work $= \int_a^b F(x)dx = \int_3^2 -\frac{q^2}{4\pi\epsilon_0 x^2}dx = \frac{q^2}{4\pi\epsilon_0}\frac{1}{x}\Big|_3^2 = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{q^2}{24\pi\epsilon_0}$

b) On the x-axis, one ion is held fixed at $x = 1$, so the distance between the ions is $x - 1$, so replace r in $F = -\frac{q^2}{4\pi\epsilon_0 r^2}$ by $x - 1$.

Work $= \int_a^b F(x)dx = \int_3^2 -\frac{q^2}{4\pi\epsilon_0(x-1)^2}dx = \frac{q^2}{4\pi\epsilon_0}\frac{1}{x-1}\Big|_3^2 = \frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{1} - \frac{1}{2}\right) = \frac{q^2}{8\pi\epsilon_0}$

c) The work can be calculated separately for each ion then added together. Work $= \frac{q^2}{24\pi\epsilon_0} + \frac{q^2}{8\pi\epsilon_0} = \frac{q^2}{6\pi\epsilon_0}$.

d) Let w be a coordinate along the rod, so $0 \le w \le 1$. Divide the rod into small pieces of width Δw. The charge in each piece is $q\Delta w$, and the force on the piece at coordinate w is $F(x) = -\frac{q\cdot q\Delta w}{4\pi\epsilon_0(x-w)^2}$. The work contributed by that piece is Work $= \int_3^2 F(x)dx = \int_3^2 -\frac{q^2\Delta w}{4\pi\epsilon_0(x-w)^2}dx = \frac{q^2\Delta w}{4\pi\epsilon_0(x-w)}\Big|_3^2 = \frac{q^2\Delta w}{4\pi\epsilon_0}\left(\frac{1}{2-w} - \frac{1}{3-w}\right)$.

We need a second integral for w from 0 to 1 to sum the work due all the pieces, and in this process we have $\Delta w \to dw$.

Total Work $= \frac{q^2}{4\pi\epsilon_0}\int_0^1\left(\frac{1}{2-w} - \frac{1}{3-w}\right)dw = \frac{q^2}{4\pi\epsilon_0}\left(-\ln(2-w) + \ln(3-w)\right)\Big|_0^1 = \frac{q^2}{4\pi\epsilon_0}(-\ln 1 + \ln 2 - (-\ln 2 + \ln 3)) = \frac{q^2}{4\pi\epsilon_0}\ln\frac{4}{3}$

Note that Work (a) $= 0.04167\frac{q^2}{\pi\epsilon_0}$, Work (b) $= 0.1667\frac{q^2}{\pi\epsilon_0}$, Work (d) $= 0.07192\frac{q^2}{\pi\epsilon_0}$, so Work (a) $<$ Work (d) $<$ Work (b).

This is reasonable because case (a) and case (b) are two extreme cases (all the charge is at one end of the rod or the other), and the total charge is the same in all three cases.

Question 10 Solution

a) $f(x) = \frac{2}{3}x^{3/2} \Rightarrow f'(x) = x^{1/2} \Rightarrow 1 + (f'(x))^2 = 1 + (x^{1/2})^2 = 1 + x$

$L = \int_a^b \sqrt{1+(f'(x))^2}dx = \int_{-1}^1 (1+x)^{1/2}dx = \frac{2}{3}(1+x)^{3/2}\Big|_{-1}^1 = \frac{2}{3}\cdot 2^{3/2} = \frac{4}{3}\sqrt{2}$

- -

b) $f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow 1 + (f'(x))^2 = 1 + \cos^2 x$

$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}dx = 2\pi \int_0^\pi \sin x(1+\cos^2 x)^{1/2}dx$, substitute $u = \cos x, du = -\sin x dx$

$S = 2\pi \int_1^{-1} (1+u^2)^{1/2} \cdot -du = 2\pi \int_{-1}^1 (1+u^2)^{1/2}du = 4\pi \int_0^1 (1+u^2)^{1/2}du$

substitute $u = \tan\theta, du = \sec^2\theta d\theta, (1+u^2)^{1/2} = \sec\theta$

$S = 4\pi \int_0^{\pi/4} \sec\theta \cdot \sec^2\theta d\theta = 4\pi \int_0^{\pi/4} \sec^3\theta d\theta = 4\pi \cdot \frac{1}{2}(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta))\Big|_0^{\pi/4}$

$= 2\pi(\sqrt{2} + \ln(\sqrt{2}+1))$

Question 11 Solution

$\bar{x} = \frac{M_y}{m}$, $\bar{y} = \frac{M_x}{m}$

case 1: $m = \int_a^b f(x)dx$, $M_x = \frac{1}{2}\int_a^b f^2(x)dx$, $M_y = \int_a^b xf(x)dx$

case 2: $m = \int_a^b (f(x)-g(x))dx$, $M_x = \frac{1}{2}\int_a^b (f^2(x)-g^2(x))dx$, $M_y = \int_a^b x(f(x)-g(x))dx$

answers : $(\bar{x},\bar{y}) =$ (a) $\left(\frac{3}{2},\frac{6}{5}\right)$, (b) $\left(\frac{3}{4},\frac{12}{5}\right)$, (c) $\left(0,\frac{2}{5}\right)$, (d) $\left(\infty,\frac{1}{4}\right)$

(b) $m = \int_0^2 (4-x^2)dx = (4x - \frac{x^3}{3})\Big|_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$

$M_x = \frac{1}{2}\int_0^2 (16-x^4)dx = \frac{1}{2}(16x - \frac{x^5}{5})\Big|_0^2 = \frac{1}{2}(32 - \frac{32}{5}) = \frac{64}{5} \Rightarrow \bar{y} = \frac{M_x}{m} = \frac{64}{5}\cdot\frac{3}{16} = \frac{12}{5}$

$M_y = \int_0^2 (4x - x^3)dx = (4\frac{x^2}{2} - \frac{x^4}{4})\Big|_0^2 = 8 - \frac{16}{4} = 4 \Rightarrow \bar{x} = \frac{M_y}{m} = 4\cdot\frac{3}{16} = \frac{3}{4}$

(d) $m = \int_0^\infty \frac{1}{1+x^2}dx$ $\left(= \arctan x\Big|_0^1 = \frac{\pi}{2} \right)$

change variable: $x = \tan\theta$, $dx = (1+\tan^2\theta)d\theta$, $x = 0 \Rightarrow \theta = 0, x = \infty \Rightarrow \theta = \frac{\pi}{2}$, since $\tan 0 = 0, \tan\frac{\pi}{2} = \infty$

$m = \int_0^\infty \frac{1}{1+x^2}dx = \int_0^{\frac{\pi}{2}} \frac{1+\tan^2\theta}{1+\tan^2\theta}d\theta = \int_0^{\frac{\pi}{2}} 1d\theta = \arctan x\Big|_0^1 = \frac{\pi}{2}$

$\bar{x} = \frac{\int_0^\infty xf(x)dx}{m} = \frac{\int_0^\infty \frac{x}{1+x^2}dx}{m} = \frac{\int_0^\infty \frac{\frac{1}{2}dx^2}{1+x^2}}{\frac{\pi}{2}} = \frac{\frac{1}{2}\ln(1+x^2)\Big|_0^\infty}{\frac{\pi}{2}} = \infty$!!!

The area is finite, but it has an infinitely large $\bar{x}$.

$\bar{y} = \frac{\int_0^\infty \frac{1}{2}f^2(x)dx}{m} = \frac{1}{m}\cdot\int_0^\infty \frac{1}{2}f^2(x)dx = \frac{1}{\frac{\pi}{2}}\int_0^\infty \frac{1}{2}\left(\frac{1}{1+x^2}\right)^2 dx = \frac{2}{\pi}\int_0^\infty \frac{\frac{1}{2}}{(1+x^2)^2}dx = \frac{2}{\pi}\int_0^\infty \frac{\frac{1}{2}+\frac{1}{2}x^2-\frac{1}{2}x^2}{(1+x^2)^2}dx$

$= \frac{2}{\pi}\left(\int_0^\infty \frac{\frac{1}{2}(1+x^2)}{(1+x^2)^2}dx - \int_0^\infty \frac{\frac{1}{2}x^2}{(1+x^2)^2}dx\right) = \frac{2}{\pi}\left(\int_0^\infty \frac{\frac{1}{2}}{1+x^2}dx - \int_0^\infty \frac{\frac{1}{2}x\cdot x}{(1+x)^2}dx\right)$

$= \frac{2}{\pi}\left(\frac{1}{2}\arctan x\Big|_0^\infty - \int_0^\infty \frac{\frac{1}{2}x\cdot\frac{1}{2}dx^2}{(1+x^2)^2}\right) = \frac{2}{\pi}\left[\frac{1}{2}\cdot\frac{\pi}{2} + \int_0^\infty \frac{x}{4}d\left(\frac{1}{1+x^2}\right)\right] = \frac{2}{\pi}\left(\frac{\pi}{4} + \frac{x}{4}\cdot\frac{1}{1+x^2}\Big|_0^\infty - \frac{1}{4}\int_0^\infty \frac{1}{1+x^2}dx\right)$

$= \frac{2}{\pi}\left(\frac{\pi}{4} + \frac{x}{4}\cdot\frac{1}{1+x^2}\Big|_0^\infty - \frac{1}{4}\int_0^\infty \frac{1}{1+x^2}dx\right) = \frac{2}{\pi}\left(\frac{\pi}{4} + \frac{x}{4}\cdot\frac{1}{1+x^2}\Big|_0^\infty - \frac{1}{4}\arctan x\Big|_0^\infty\right)$

$= \frac{2}{\pi}\left(\frac{\pi}{4} + 0 - \frac{1}{4}\cdot\frac{\pi}{2}\right) = \frac{2}{\pi}\cdot\frac{\pi}{8} = \frac{1}{4}$

Question 12 Solution

$f(t) = ce^{-ct}$, where $c = \frac{1}{1000}$ and $t \geq 0$

a) $\text{Prob}(0 \leq t \leq 200) = \int_0^{200} ce^{-ct}dt = -e^{-ct}\big|_0^{200} = 1 - e^{-\frac{1}{5}} \approx 0.18$

b) $\text{Prob}(t \geq 800) = \int_{800}^\infty ce^{-ct}dt = -e^{-ct}\big|_{800}^\infty = e^{-\frac{4}{5}} \approx 0.45$

Question 13 Solution

Note that $f(x) \geq 0$ for all x, so we need only show that $\int_0^1 f(x)dx = 1$.

First complete the square.

$x(1-x) = x - x^2 = -x^2 + x = -(x^2 - x) = -(x^2 - x + \frac{1}{4} - \frac{1}{4}) = -((x-\frac{1}{2})^2 - \frac{1}{4}) = \frac{1}{4} - (x-\frac{1}{2})^2$

Then substitute $u = x - \frac{1}{2}, du = dx$, which gives $\int_0^1 f(x)dx = \int_0^1 \frac{1}{\pi\sqrt{x(1-x)}}dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\pi\sqrt{\frac{1}{4}-u^2}}du.$

Then substitute $u = \frac{1}{2}\sin\theta$, which gives $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\pi\sqrt{\frac{1}{4}-u^2}}du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi\cdot\frac{1}{2}\cos\theta}\cdot\frac{1}{2}\cos\theta d\theta = \frac{1}{\pi}\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1.$

Differential Equations

Question 14 Solution

a) $y' = -2y$, $y_0 = 1$ (standard exponential decay) $\Rightarrow \quad y = C \cdot e^{-2t}$.

$y_0 = 1 \;\Rightarrow\; C = 1 \quad \Rightarrow \quad y(t) = e^{-2t}$ and $\lim\limits_{t\to\infty} y(t) = 0$

- -

b) $y' = 1 - 2y \quad \Rightarrow \quad \frac{dy}{dt} = 1 - 2y$ separation of variables $\quad \Rightarrow \quad \frac{dy}{1-2y} = dt$ integrate both sides $\Rightarrow \quad -\frac{1}{2}\ln|1-2y| = t + C \quad \Rightarrow \quad 1 - 2y = C \cdot e^{-2t} \quad \Rightarrow \quad y = \frac{1-C\cdot e^{-2t}}{2}$

$y_0 = 0 \quad \Rightarrow \quad C = 1 \quad \Rightarrow \quad y = \frac{1-e^{-2t}}{2}$ and $\lim\limits_{t\to\infty} y(t) = \frac{1}{2}$

- -

c) $y' = 1 - y^2 \quad \Rightarrow \quad \frac{dy}{dt} = (1+y)(1-y)$ separation of variables $\quad \Rightarrow \quad \frac{dy}{(1+y)(1-y)} = dt$ partial fraction $\quad \Rightarrow \quad \frac{\frac{1}{2}}{1+y}dy + \frac{\frac{1}{2}}{1-y}dy = dt$ integrate both sides $\quad \Rightarrow \quad \frac{1}{2}\ln|1+y| - \frac{1}{2}\ln|1-y| = t + C$ $\quad \Rightarrow \quad \ln\left|\frac{1+y}{1-y}\right| = 2t + C \quad \Rightarrow \quad \frac{1+y}{1-y} = C \cdot e^{2t}$ where C is constant may be positive or negative $\quad \Rightarrow$ $y(t) = \frac{C\cdot e^{2t}-1}{C\cdot e^{2t}+1}$

$y_0 = 0 \;\Rightarrow\; C = 1 \quad \Rightarrow \quad y(t) = \frac{e^{2t}-1}{e^{2t}+1}$. Furthermore $y(t) = \frac{(e^{2t}-1)e^{-t}}{(e^{2t}+1)e^{-t}} = \frac{e^{t}-e^{-t}}{e^{t}+e^{-t}} = \tanh t$

Check... $y(t) = \tanh t$ is the solution.

$\lim\limits_{t\to\infty} y(t) = 1$

- -

d) $y' = -ty \quad \Rightarrow \quad \frac{dy}{dt} = -ty$ separation of variables $\quad \Rightarrow \quad \frac{dy}{y} = -tdt$ integrate both sides $\quad \Rightarrow$ $\ln|y| = -\frac{1}{2}t^2 + C \quad \Rightarrow \quad y = Ce^{-\frac{1}{2}t^2}$

$y_0 = 1 \quad \Rightarrow \quad C = 1 \quad \Rightarrow \quad y(t) = e^{-\frac{1}{2}t^2}$

$\lim\limits_{t\to\infty} y(t) = 0$

Question 15 Solution

a) $y = c_1e^t + c_2e^{-t} \;\Rightarrow\; y' = c_1e^t - c_2e^{-t} \;\Rightarrow\; y'' = c_1e^t + c_2e^{-t} = y$

Thus y is a solution of $y'' = y$ for any constants c_1, c_2.

b) $y(0) = 1, y'(0) = 0 \Rightarrow c_1 + c_2 = 1,\; c_1 - c_2 = 0 \Rightarrow c_1 = c_2 = \frac{1}{2} \Rightarrow y(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh x$

c) $y(0) = 0, y'(0) = 1 \Rightarrow c_1 + c_2 = 0,\; c_1 - c_2 = 1 \Rightarrow c_1 = \frac{1}{2}, c_2 = -\frac{1}{2} \Rightarrow y(t) = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh x$

Question 16 Solution

The cell count in a bacteria culture grows at a rate proportional to its size. After 30 minutes there are 200 cells and after 90 minutes there are 800 cells. (a) Find the initial cell count. (b) When will the cell count reach 6400? The answers should be expressed as integers.

a) Let $y(t)$ be the cell count after t minutes. The cell count satisfies the differential equation $y' = ky$, with solution $y(t) = y_0e^{kt}$. We have $y(30) = y_0e^{30k} = 200$ and $y(90) = y_0e^{90k} = 800$. Dividing the 2nd equation by the 1st equation yields $e^{60k} = 4$, which implies $e^{30k} = 2$. Then substituting into the 1st equation yields $y_0 \cdot 2 = 200$ and hence $y_0 = 100$ cells.

b) From $e^{30k} = 2$, we obtain $k = \frac{\ln 2}{30}$, hence $y(t) = y_0e^{kt} = 100e^{(\ln 2/30)t} = 100e^{(\ln 2)\cdot t/30} = 100 \cdot 2^{t/30}$. Then we set $y(t) = 100 \cdot 2^{t/30} = 6400 = 100 \cdot 2^6$, hence $t/30 = 6$, and finally $t = 180$ hours.

Question 17 Solution

$y(t) = y_0e^{-kt}$

$y(t) = 40 \cdot \left(\frac{1}{2}\right)^{\frac{t}{1.4\times10^{-4}}}$

$30 = 40 \cdot \left(\frac{1}{2}\right)^{\frac{t}{1.4\times10^{-4}}}$

$t = 1.4 \times 10^{-4}\frac{\ln\frac{3}{4}}{\ln\frac{1}{2}} = 0.581 \times 10^{-4}\text{s}$

Question 18 Solution

$y(t)$: tiger body mass (kg) as a function of time t (day)
$y' = \text{rate in} - \text{rate out} = 2500\frac{\text{cal}}{\text{day}} \cdot \frac{1\,\text{kg}}{10000\,\text{cal}} - 20\,\text{cal} \cdot \frac{y\,\text{kg}}{10000\,\text{cal}} \cdot \frac{1}{\text{day}} = \frac{2500-20y}{10000}\frac{\text{kg}}{\text{day}}$
$y' = -\frac{1}{500}(y-125)$: Newton's heating/cooling, $y' = k(y-T)$
$y(t) = T + (y_0 - T)e^{kt} = 125 + (y_0 - 125)e^{-\frac{t}{500}} \Rightarrow \lim_{t\to\infty} y(t) = 125$ kg

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Question 19 Solution

$y(t) = T + (y_0 - T)e^{-kt}$, note that the patient's temperature is T , $y_0 = 70°$F
$y(1) = 95 = T + (70-T)e^{-k}$
$y(2) = 100 = T + (70-T)e^{-2k}$
$\Rightarrow (\frac{95-T}{70-T})^2 = \frac{100-T}{70-T} \Rightarrow (95-T)^2 = (100-T)(70-T) \Rightarrow T^2 - 190T + 95^2 = T^2 - 170T + 70000 \Rightarrow 20T = 2025$
$\Rightarrow T = 101.25°$ F

Question 20 Solution

$y' = ky(M-y)$
$y(t) = \frac{My_0}{y_0 + (M-y_0)e^{-kMt}}$
$y_0 = 10$, $M = 4000$
$y(t) = \frac{40000}{10 + (4000-10)e^{-4000kt}}$
measure time in days
$20 = \frac{40000}{10 + (4000-10)e^{-4000\cdot 7k}}$
$e^{-k} = \left(\frac{199}{399}\right)^{\frac{1}{28000}}$
$y(t) = \frac{40000}{10 + 3990\left(\frac{199}{399}\right)^{\frac{t}{7}}}$
let $y(t) = \frac{1}{2}\cdot 4000 = 2000$, solve $t = \frac{7\ln 399}{\ln 399 - \ln 199} \approx 60$ days.

Question 21 Solution

This equation, $y' = f(y)$ with $f(y) = 2y$, is particularly simple to study with Euler's Method. In this case we have $u_{k+1} = u_k + \Delta t \cdot f(u_k) = u_k + \Delta t \cdot 2u_k = (1+2\Delta t)u_k$, which yields $u_k = (1+2\Delta t)^k u_0 = (1+2\Delta t)^k$. So $u_n = (1+2/n)^n$, where $n = 1/\Delta t$ is the number of steps.

Δt	u_n
1	$(1+2)^1 = 3$
1/2	$(1+1)^2 = 4$
1/4	$(1+1/2)^4 = 5.0625$

The general solution of this differential equation is $y = Ce^{2t}$, and the solution which fits our initial condition is $y = e^{2t}$. Thus the true answer is $y(1) = e^2 \approx 7.389$. We need to use smaller steps to get a satisfactory approximation. With 1000 steps, we would still only have $u_n \approx 7.374$.

Series

Question 22 Solution

a) divergent $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ by p-test of series, $p = 1$.
b) convergent since $\sum_{n=1}^{\infty}\frac{1}{2^n} = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{2}} = 1 < \infty$.
c) convergent by p-test of series, $p = 2$.
d) convergent by Alternating Series Test, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n^2} = 0$, $a_{n+1} < a_n$ and the sign is alternating.
e) divergent by Ratio Test, $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)^2}\cdot\frac{n^2}{2^n} = 2 > 1$, ($L > 1$ divergent)

Question 23 Solution

a) $0.111111.... = 0.1 + 0.01 + 0.001 + 0.0001 + \cdots = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{1000} + \cdots = \sum_{n=1}^{\infty} \frac{1}{10^n} = \frac{1}{10} \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = \frac{1}{10} \cdot \frac{1}{1-\frac{1}{10}} = \frac{1}{9}$

b) $0.1212121212... = \frac{12}{100} + \frac{12}{10000} + \frac{12}{1000000} + \cdots = \sum_{n=1}^{\infty} \frac{12}{100^n} = \frac{12}{100} \sum_{n=0}^{\infty} \frac{1}{100^n} = \frac{12}{100} \cdot \frac{1}{1-\frac{1}{100}} = \frac{12}{99}$

c) $0.4999999... = 0.45 + 0.045 + 0.0045 + 0.00045 + \cdots = \frac{45}{100} + \frac{45}{1000} + \frac{45}{10000} + \cdots$

$= \frac{45}{100}\left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots\right) = \frac{45}{100} \sum_{n=0}^{\infty} \frac{1}{10^n} = \frac{45}{100} \cdot \frac{1}{1-\frac{1}{10}} = \frac{1}{2}$ (ie, $0.49999999... = 0.5$)

Question 24 Solution

a) recall : $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ for all x, so setting $x = 2$ we obtain $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$

b) recall : $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $-1 < x < 1$, so setting $x = \frac{1}{3}$ we obtain $\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} \frac{1}{3^n} - 1 = \frac{3}{2} - 1 = \frac{1}{2}$

c) differentiating the first equation in part (b) with respect to x we obtain $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$ for $-1 < x < 1$, so setting $x = \frac{1}{3}$ we obtain $\sum_{n=1}^{\infty} n(\frac{1}{3})^{n-1} = \frac{1}{(1-\frac{1}{3})^2} = \frac{9}{4} \Rightarrow \sum_{n=1}^{\infty} n(\frac{1}{3})^n \cdot (\frac{1}{3})^{-1} = \frac{9}{4} \Rightarrow \sum_{n=1}^{\infty} n(\frac{1}{3})^n = \frac{9}{4} \cdot \frac{1}{3} = \frac{3}{4}$

Question 25 Solution

Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, note that $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ includes all the odd terms. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ =odd terms+even terms $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\Rightarrow \frac{\pi^2}{6} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \frac{1}{4}\frac{\pi^2}{6} \Rightarrow \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}$.

Question 26 Solution

a) Use $|s - s_{10}| \leq \int_n^{\infty} f(x)dx$ since all terms are positive, where $f(x) = \frac{1}{x^2}$

$|s - s_{10}| \leq \int_{10}^{\infty} f(x)dx = \int_{10}^{\infty} \frac{1}{x^2}dx = -\frac{1}{x}\Big|_{10}^{\infty} = 0.1$

b) Use $|s - s_{10}| \leq a_{n+1}$ where $a_{n+1} = \frac{1}{(n+1)^2}$ since the series is an alternating series.

$|s - s_{10}| \leq a_{n+1} = \frac{1}{11^2} = \frac{1}{121}$

Question 27 Solution

Two students walk towards each other at 2 mi/hr starting from a separation of 20 miles. At the same time, a dog starts running back and forth between the students at 10 mi/hr. Let D be the total distance the dog has traveled when the students finally meet. Express D as an infinite series and find the sum of the series.

Assume the dog starts with student A and runs to student B; if the time interval has duration t_0, then we have $10t_0 + 2t_0 = 20$ (the distance the dog runs plus the distance student B walks is equal to the distance between them when they start). This implies that $12t_0 = 20$ or $t_0 = \frac{20}{12} = \frac{5}{3}$.

Note that at the end of the first interval, the distance between the students is $20 - 4t_0 = 20 - 4 \cdot \frac{5}{3} = 20 - \frac{20}{3} = \frac{40}{3} = 20 \cdot \frac{2}{3}$; in other words the distance between the students decreased by a factor of $\frac{2}{3}$.

In the next time interval of duration t_1, the dog runs back to A; then we have $10t_1 + 2t_1 = \frac{40}{3}$ (the distance the dog runs plus the distance A walks is equal to the distance between the students when they start).

This implies that $12t_1 = \frac{40}{3}$ or $t_1 = \frac{40}{12 \cdot 3} = \frac{10}{9} = \frac{5}{3} \cdot \frac{2}{3}$.

Note that at the end of the second interval, the distance between the students is $\frac{40}{3} - 4 \cdot t_1 = \frac{40}{3} - 4 \cdot \frac{10}{9} = \frac{80}{9} = \frac{40}{3} \cdot \frac{2}{3}$.
In the next time interval of duration t_2, the dog runs back to B; then we have $10t_2 + 2t_2 = \frac{80}{9}$ (the distance the dog runs plus the distance B walks is equal to the distance between them when they start).
This implies that $12t_2 = \frac{80}{9}$ or $t_2 = \frac{80}{12 \cdot 9} = \frac{20}{27} = \frac{5}{3} \cdot (\frac{2}{3})^2$.
The pattern repeats.
The total time is $T = t_0 + t_1 + t_2 + \cdots = \frac{5}{3} + \frac{5}{3} \cdot \frac{2}{3} + \frac{5}{3} \cdot \left(\frac{2}{3}\right)^2 + \cdots = \frac{5}{3} \cdot \left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right) = \frac{5}{3} \cdot \frac{1}{1-\frac{2}{3}} = 5$.
Hence the dog travels a distance $D = 10$ mph $\times$ 5 hours $= 50$ miles.
The question asks us to express D as an infinite series, but we can also find the distance directly as follows. The students meet in the middle after walking a distance of 10 miles; since they walk at 2 mph, this must take 5 hours. This is also how long the dog runs, so the dog must run a total distance of 10 mph $\times$ 5 hours $=$ 50 miles.

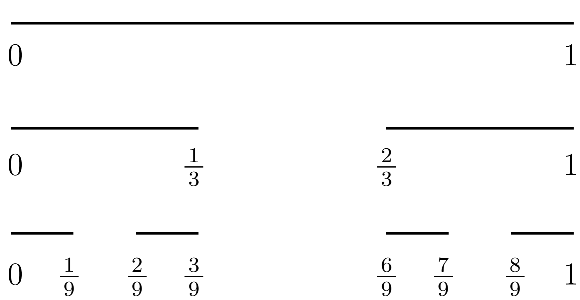
Question 28 Solution

Consider the remaining points as happening in pairs. In any pair of points, there are 3 possible outcomes; (a) you win both, with probability p^2, (b) you lose both, with probability $(1-p)^2$, or (c) you win one and lose one, with probability $2p(1-p)$. In the first case, the game ends and you win. In the second case, the game ends and you lose. In the third case, the game continues. So any scenario in which you win has the following form; you split pairs of points with your opponent a certain number of times and then you win a pair. Thus the probability of winning is $p^2 + (2p(1-p))p^2 + (2p(1-p))^2p^2 + \cdots = \sum_{n=0}^{\infty} [2p(1-p)]^n p^2 = p^2 \sum_{n=0}^{\infty} [2p(1-p)]^n = p^2 \cdot \frac{1}{1-2p(1-p)} = \frac{p^2}{1-2p+2p^2}$.

$p = \frac{1}{2} \;\Rightarrow\; \frac{p^2}{1-2p+2p^2} = \frac{1}{2}$ (no surprise)
$p = \frac{1}{4} \;\Rightarrow\; \frac{p^2}{1-2p+2p^2} = \frac{1}{10}$
$p = \frac{3}{4} \;\Rightarrow\; \frac{p^2}{1-2p+2p^2} = \frac{9}{10}$ (no surprise that this answer and the previous would sum to 1)

Question 29 Solution

Start with the closed interval $[0,1]$; remove the open interval $(\frac{1}{3}, \frac{2}{3})$; that leaves the two closed intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$; remove the middle third of those; that leaves four closed intervals; remove the middle third of those; continue the process indefinitely. The Cantor set is the set of all points remaining after all the open intervals have been removed. (a) Sketch the remaining closed intervals for the first three steps. (b) Show that the total length of all the open intervals removed is 1. (c) Show that, nonetheless, the Cantor set contains infinitely many points.

a)


$[0,1]$

$[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

b) total length removed $= \frac{1}{3} + 2 \cdot \frac{1}{3} \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} + \cdots = \frac{1}{3}\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right] = \frac{1}{3} \cdot \frac{1}{1-\frac{2}{3}} = 1$
c) Any endpoint of a remaining closed interval is never be removed; every step doubles the number of remaining intervals; this yields infinitely many points. Some points in the Cantor are $\{0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}\}$. Can you find a few more?

Power Series, Taylor Series

Question 30 Solution

a) $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = |x| < 1 \;\Rightarrow\;$ the radius of convergence is 1; since at two end points $x = \pm 1$, the series diverges, the interval of convergence is $-1 < x < 1$. The sum is $\frac{1}{1-x}$ for $-1 < x < 1$.

b) $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^n x^{n+1}}{2^{n+1}x^n}\right| = \left|\frac{x}{2}\right| < 1 \;\Rightarrow\; |x| < 2 \;\Rightarrow\;$ the radius of convergence is 2; since at two end points $x = \pm 2$, the series diverges, the interval of convergence is $-2 < x < 2$. The sum is $\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \frac{1}{1-\frac{x}{2}} = \frac{2}{2-x}$ for $-2 < x < 2$.

c) $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(x-1)^n}\right| = |x-1| < 1 \;\Rightarrow\; -1 < x-1 < 1 \Rightarrow 0 < x < 2 \;\Rightarrow\;$ the radius of convergence is 1 (the length of the interval divided by 2); since at two end points $x = 0$ and 2, the series diverges, the interval of convergence is $0 < x < 2$. The sum is $\frac{1}{1-(x-1)} = \frac{1}{2-x}$ for $0 < x < 2$.

d) $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{nx^{n+1}}{(n+1)x^n}\right| = |x| < 1 \;\Rightarrow\; -1 < x < 1 \;\Rightarrow\;$ the radius of convergence is 1 ; since at $x = 1$, the series is harmonic series thus diverges, while at $x = -1$, the series converges by AST (alternating series test), the interval of convergence is $-1 \le x < 1$. Note that $x^n = \int nx^{n-1}dx \;\Rightarrow\; \frac{x^n}{n} = \int x^{n-1}dx$ the sum is $\sum_{n=1}^{\infty}\int x^{n-1}dx = \int\sum_{n=1}^{\infty}x^{n-1}dx = \int\sum_{n=0}^{\infty}x^n dx = \int\frac{1}{1-x}dx = -\ln(1-x) = \ln\frac{1}{1-x}$ for $-1 \le x < 1$.

$\ln\frac{1}{1-x} = \sum_{n=1}^{\infty}\frac{x^n}{n}$

e) $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{nx^n}\right| = |x| < 1 \;\Rightarrow\; -1 < x < 1 \;\Rightarrow\;$ the radius of convergence is 1; since at $x = \pm 1$, the series diverges, the interval of convergence is $-1 < x < 1$.

Note that $(x^n)' = nx^{n-1}$, $\sum_{n=1}^{\infty}nx^n = x\sum_{n=1}^{\infty}nx^{n-1} = x\sum_{n=1}^{\infty}(x^n)' = x\cdot\left(\sum_{n=0}^{\infty}x^n\right)' = x\cdot\left(\frac{1}{1-x}\right)' = x\cdot\frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}$

$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty}nx^n$

Question 31 Solution

We need to find c_n such that $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty}c_n(x-\frac{1}{2})^n$.

$$\frac{1}{1-x} = \frac{1}{1-(x-\frac{1}{2}+\frac{1}{2})} = \frac{1}{1-(x-\frac{1}{2})-\frac{1}{2}} = \frac{1}{\frac{1}{2}-(x-\frac{1}{2})} = \frac{1}{\frac{1}{2}(1-2(x-\frac{1}{2}))} = 2\sum_{n=0}^{\infty}(2(x-\tfrac{1}{2}))^n = \sum_{n=0}^{\infty}2^{n+1}(x-\tfrac{1}{2})^n$$

Question 32 Solution

$$\begin{aligned}
&f(x) = \sinh x \;\Rightarrow\; f(0) = \sinh 0 = 0\\
&f'(x) = \cosh x \;\Rightarrow\; f'(0) = \cosh 0 = 1\\
&f''(x) = \sinh x \;\Rightarrow\; f''(0) = \sinh 0 = 0\\
&f'''(x) = \cosh x \;\Rightarrow\; f'''(0) = \cosh 0 = 1\\
&\vdots
\end{aligned}$$

$$\sinh x = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5\cdots = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

$$\cosh x = (\sinh x)' = \left(\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\right)' = \sum_{n=0}^{\infty}\left(\frac{x^{2n+1}}{(2n+1)!}\right)' = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

Question 33 Solution

a) $\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, $\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$\sin^2 x + \cos^2 x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^2$$
$$= \left(x^2 - 2\frac{x^4}{3!} + \frac{x^6}{36} + 2\frac{x^6}{5!} + \cdots\right) + \left(1 - 2\frac{x^2}{2} + \frac{x^4}{4} + 2\frac{x^4}{4!} - 2\frac{x^6}{2\cdot 4!} - 2\frac{x^6}{6!} + \cdots\right) = \cdots = 1 + O(x^8)$$

b) In fact we know that all terms in the power series for $\sin^2 x + \cos^2 x$ vanish after the first term 1, though proving it is a nice fairly involved exercise.

Question 34 Solution

Because $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \cdots$

$T_1(x) = T_0(x) = 1$ and $T_2(x) = 1 - x^2$. See Figure below for sketch.

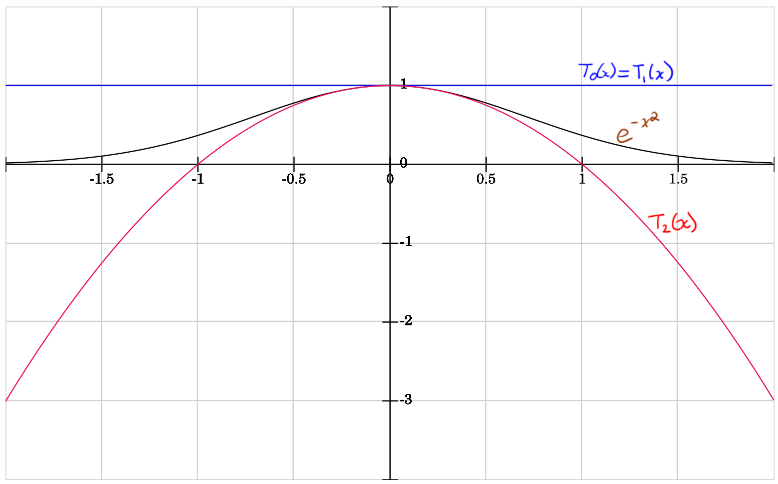


Figure 1: Graph for Problem 34.

Question 35 Solution

Show that $0 \le f(x) < 1$; $\lim_{x\to\infty} f(x) = 1$; $\lim_{x\to 0^+} f^{(n)}(x) = \lim_{x\to 0^+} P(\frac{1}{x})e^{-1/x}$, where $P(\frac{1}{x})$ is a polynomial of $\frac{1}{x}$; when $x \to 0^+$ $e^{-1/x} \to 0$ exponentially (faster than any polynomial) thus $f^{(n)}(x) \to 0$ regardless of the form of $P(\frac{1}{x})$.

Question 36 Solution

method 1 It can be shown using Taylor series for $f(x) = \sqrt{x}$ about $x = a$, that $\sqrt{x} = \sqrt{a} + \frac{1}{2\sqrt{a}}(x - a) - \frac{1}{8}a^{-\frac{3}{2}}(x-a)^2 + \cdots$.

Setting $a = 9$ yields $\sqrt{x} = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \cdots$.

This is a convergent alternating series (why?), so $|s - s_n| < a_{n+1}$, i.e. $\left|\sqrt{x} - \left(3 + \frac{1}{6}(x-9)\right)\right| < \frac{1}{216}(x-9)^2$.

Setting $x = 10$ yields $\left|\sqrt{10} - 3.16666\right| < \frac{1}{216} < \frac{1}{200} = 0.005$.

The approximate value is $\sqrt{10} \approx 3.16666$.

method 2 Use the binomial series, $(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \cdots$ for $-1 < x < 1$.

$\sqrt{10} = \sqrt{9+1} = \sqrt{9(1+\frac{1}{9})} = \sqrt{9}\sqrt{1+\frac{1}{9}} = 3(1+\frac{1}{9})^{1/2}$; we have $x = \frac{1}{9}$ and $k = \frac{1}{2}$

$\sqrt{10} = 3(1 + \frac{1}{2}\cdot\frac{1}{9} - \frac{1}{8}\cdot\frac{1}{9^2} + \cdots) = 3 + \frac{1}{6} - \frac{1}{216} + \cdots \Rightarrow \left|\sqrt{10} - 3.16666\right| < \frac{1}{216} < \frac{1}{200} = 0.005$

Question 37 Solution

Use the Taylor series for $f(x) = \ln(1+x)$ about $x = 0$ to evaluate $\ln\frac{3}{2}$ to within 10^{-3}.

In class we showed that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$ for $-1 < x < 1$.

For $x = \frac{1}{2}$, this gives $\ln\frac{3}{2} = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n2^n} = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots$. This a convergent alternating series, so we can use the error bound for alternating series, $|s - s_n| \le a_{n+1}$, where s is the exact

sum of the series, s_n is the nth partial sum, and $a_{n+1} = \frac{1}{(n+1)2^{n+1}}$ is the first neglected term.
We search for the smallest value of n such that $a_{n+1} = \frac{1}{(n+1)2^{n+1}} \le 10^{-3}$. We find that $n = 7$, where $a_8 = \frac{1}{8 \cdot 2^8} = 0.00048828$, and therefore we take $s_7 = 0.4058$ as the approximation; the exact value is $s = \ln \frac{3}{2} = 0.4055$, so the error is less than 10^{-3}, as required. Note however that $s_6 = 0.4047$, which also has error less than 10^{-3}; this shows that the error bound is not optimal.

Question 38 Solution

Find the first two nonzero terms in the Taylor series for $f(x)$ about $x = 0$.

a) $e^{-x} \sin x = x - x^2 + \cdots$

recall that $e^x = 1 + x + \frac{1}{2}x^2 + \cdots$, so we have $e^{-x} = 1 - x + \frac{1}{2}x^2 + \cdots$, also recall that $\sin x = x - \frac{1}{6}x^3 + \cdots$

then we have $e^{-x} \sin x = (1 - x + \frac{1}{2}x^2 + \cdots)(x - \frac{1}{6}x^3 + \cdots) = x - x^2 + \cdots$ ok

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b) $\frac{1-\cos x}{x} = \frac{1}{2}x - \frac{1}{24}x^3 + \cdots$

recall that $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots$, then $\frac{1-\cos x}{x} = \frac{1-(1-\frac{1}{2}x^2+\frac{1}{24}x^4+\cdots)}{x} = \frac{1}{2}x - \frac{1}{24}x^3 + \cdots$ ok

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c) $\tan x = x + \frac{1}{3}x^3 + \cdots$

recall that $\sin x = x - \frac{1}{6}x^3 + \cdots$, $\cos x = 1 - \frac{1}{2}x^2 + \cdots$

then $\tan x = \frac{\sin x}{\cos x} = \frac{x-\frac{1}{6}x^3+\cdots}{1-\frac{1}{2}x^2+\cdots} = (x - \frac{1}{6}x^3 + \cdots) \cdot \frac{1}{1-(\frac{1}{2}x^2+\cdots)}$: write 2nd factor as geometric series

$= (x - \frac{1}{6}x^3 + \cdots) \cdot (1 + (\frac{1}{2}x^2 + \cdots) + \cdots) = x + x^3(\frac{1}{2} - \frac{1}{6}) + \cdots = x + \frac{1}{3}x^3 + \cdots$ ok

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d) $\tan^{-1}x = x - \frac{1}{3}x^3 + \cdots$

$\tan^{-1}x = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$, note that $\tan^{-1} 0 = 0$, so we must have $c_0 = 0$

$\Rightarrow \tan^{-1}x = c_1x + c_2x^2 + c_3x^3 + \cdots = y$

$\Rightarrow x = \tan y = y + \frac{1}{3}y^3 + \cdots$: this uses part (c)

$= (c_1x + c_2x^2 + \cdots) + \frac{1}{3}(c_1x + c_2x^2 + \cdots)^3 + \cdots = c_1x + c_2x^2 + c_3x^3 + \cdots + \frac{1}{3}(c_1^3x^3 + \cdots) + \cdots$

we've shown that $x = c_1x + c_2x^2 + c_3x^3 + \cdots + \frac{1}{3}(c_1^3x^3 + \cdots) + \cdots$

the coefficients of each power of x must match on the left and right

$x^1 \Rightarrow 1 = c_1$

$x^2 \Rightarrow 0 = c_2$

$x^3 \Rightarrow 0 = c_3 + \frac{1}{3}c_1^3 \Rightarrow c_3 = -\frac{1}{3}c_1^3 = -\frac{1}{3}$ ok

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Question 39 Solution

$B_0 = f(0) = 1$; $B_1 = f'(0) = -\frac{1}{2}$; $B_2 = f''(0) = \frac{1}{6}$ (using L'Hospital rule).

Question 40 Solution

a) $f(x) = x, f(0) = 0, f'(0) = 1$
b) $f(x) = \sin x, f(0) = 0, f'(x) = \cos x, f'(0) = 1$
c) $f(x) = \ln(1 + x), f(0) = 0, f'(x) = \frac{1}{1+x}, f'(0) = 1$
b) $f(x) = e^x - 1, f(0) = 0, f'(x) = e^x, f'(0) = 1$

If the functions are sketched in a neighborhood of $x = 0$, the order they appear (from top to bottom), consider their Taylor approximations

$x = x$, $\sin x = x - \frac{1}{6}x^3 + \cdots$, $\ln(1 + x) = x - \frac{1}{2}x^2 + \cdots$, $e^x - 1 = x + \frac{1}{2}x^2$

Thus on right hand side of 0, from top to bottom, $e^x - 1$, x, $\sin x$ and $\ln(1 + x)$; on left hand side of 0, from top to bottom, $e^x - 1$, $\sin x$, x, and $\ln(1 + x)$.

Question 41 Solution

a) $J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots$ it is alternating series.

$\int_0^1 \left(1 - \frac{x^2}{4}\right) dx = \frac{11}{12}$

error bound $\int_0^1 \frac{x^4}{64} dx = \frac{x^5}{5 \cdot 64} = \frac{1}{320}$

b) $J_0(x)' = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)x^{2n+1}}{2^{2n+2}((n+1)!)^2}$

$J_0(x)'' = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n-2}}{2^{2n}(n!)^2}$

$xJ_0(x)'' = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n-1}}{2^{2n}(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n+1}}{2^{2n+2}((n+1)!)^2}$

$xJ_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n}(n!)^2}$

$xJ_0(x)'' + J_0(x)' + xJ_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{2n+1}}{2^{2n}(n!)^2} \left[\frac{2n+2}{2^2(n+1)^2} + \frac{(2n+1)(2n+2)}{2^2(n+1)^2} - 1\right] = 0$

Thus $J_0(x)$ satisfies $xy'' + y' + xy = 0$

Question 42 Solution

$f(t) = \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + t^3 + t^4 + \cdots$

a) step 1 : differentiate the series, $f'(t) = 1 + 2t + 3t^2 + 4t^3 + \cdots$

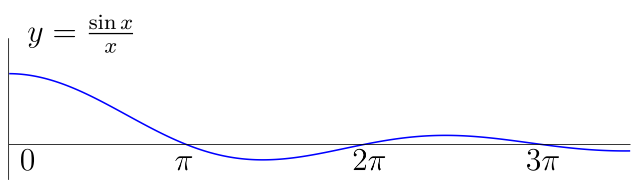
step 2 : square the series, $f^2(t) = (1+t+t^2+t^3+t^4+\cdots)\cdot(1+t+t^2+t^3+t^4+\cdots) = 1+2t+3t^2+4t^3+\cdots$

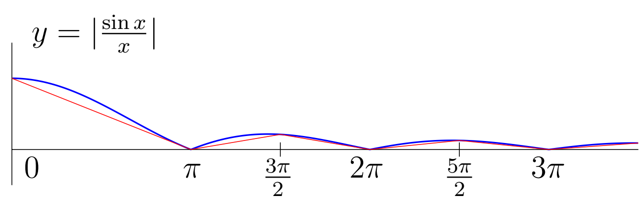
Thus $f'(t) = f^2(t)$, so $f(t)$ satisfies the differential equation $y' = y^2$ and we can check that the initial condition is $f(0) = 1$.

b) $y' = y^2 \Rightarrow \frac{dy}{dt} = y^2 \Rightarrow \frac{dy}{y^2} = dt \Rightarrow \int \frac{dy}{y^2} = \int dt \Rightarrow -\frac{1}{y} = t + C \Rightarrow y = \frac{1}{-C-t}$, now apply initial condition $t = 0, y = 1 \Rightarrow C = -1 \Rightarrow y = \frac{1}{1-t}$

Since we showed in part (a) that $f(t)$ satisfes the differential equation and initial condition, we obtain $f(t) = \frac{1}{1-t}$. This is a round-about way of deriving the sum of a geomertic series, $f(t) = \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + t^3 + +t^4 + \cdots = \frac{1}{1-t}$.

Question 43 Solution





a1) $\int_0^{\infty} \frac{\sin x}{x} dx$ converges

pf: $\int_0^{\infty} \frac{\sin x}{x} dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{\sin x}{x} dx = \sum_{n=0}^{\infty} (-1)^n a_n$, where $a_n = \int_{n\pi}^{(n+1)\pi} |\frac{\sin x}{x}| dx$,

the series is an alternating series; we will check the 3 conditions of the AST

(1) $a_n > 0$: this is obvious

(2) $a_{n+1} < a_n$: substitute $x = n\pi + u, dx = du$, then $a_n = \int_{n\pi}^{(n+1)} |\frac{\sin x}{x}| dx = \int_0^{\pi} |\frac{\sin(n\pi+u)}{n\pi+u}| du$

$= \int_0^{\pi} |\frac{\sin n\pi \cos u + \cos n\pi \sin u)}{n\pi+u}| du = \int_0^{\pi} \frac{\sin u}{n\pi+u} du$, where we used $\sin n\pi = 0, \cos n\pi = (-1)^n$

then we have $a_{n+1} = \int_0^{\pi} \frac{\sin u}{(n+1)\pi+u} du < \int_0^{\pi} \frac{\sin u}{n\pi+u} du = a_n$

(3) $\lim_{n\to\infty} a_n = 0$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \int_0^{\pi} \frac{\sin u}{n\pi+u} du = \int_0^{\pi} \lim_{n\to\infty} \frac{\sin u}{n\pi+u} du = \int_0^{\pi} 0 \, du = 0$ ok

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a2) $\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx$ diverges

pf: $\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx = \int_0^{\pi} \left|\frac{\sin x}{x}\right| dx + \int_{\pi}^{2\pi} \left|\frac{\sin x}{x}\right| dx + \int_{2\pi}^{3\pi} \left|\frac{\sin x}{x}\right| dx + \int_{3\pi}^{4\pi} \left|\frac{\sin x}{x}\right| dx + \cdots$

the area under each curve can be bounded below by the area of a triangle

$\int_0^\infty |\frac{\sin x}{x}|dx > \frac{1}{2}\cdot\pi\cdot 1 + \frac{1}{2}\cdot\pi\cdot\frac{1}{3\pi/2} + \frac{1}{2}\cdot\pi\cdot\frac{1}{5\pi/2} + \frac{1}{2}\cdot\pi\cdot\frac{1}{7\pi/2} + \cdots > 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \sum_{n=1}^{\infty}\frac{1}{2n-1}$

The series has the form $\sum_{n=1}^{\infty} a_n$, where $a_n = f(n), f(x) = \frac{1}{2x-1}$, and $f(x)$ is positive and decreasing for $x \geq 1$. Hence the integral test can be applied; $\int_1^\infty f(x)dx = \int_1^\infty \frac{dx}{2x-1} = \frac{1}{2}\ln(2x-1)\Big|_1^\infty = \frac{1}{2}\ln\infty - \frac{1}{2}\ln 1 = \infty$. ok

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b) $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$

pf: set $f(a) = \int_0^\infty \frac{\sin x}{x}e^{-ax}dx$; then note that $f(0) = \int_0^\infty \frac{\sin x}{x}dx$; so we need to evaluate $f(0)$; to do that we will evaluate $f(a)$ and then set $a = 0$; to solve for $f(a)$, we will find an expression for $f'(a)$ and then integrate; hence we have $f'(a) = \int_0^\infty \frac{\sin x}{x}\frac{d}{da}[e^{-ax}]dx = \int_0^\infty \frac{\sin x}{x}\cdot -xe^{-ax}dx = -\int_0^\infty \sin x e^{-ax}dx$

substitute: $u = \sin x, dv = e^{-ax}dx \Rightarrow du = \cos x dx, v = \frac{e^{-ax}}{-a}$

$\Rightarrow f'(a) = -\left[\sin x\cdot\frac{e^{-ax}}{-a}\Big|_0^\infty - \int_0^\infty \cos x\frac{e^{-ax}}{-a}dx\right] = -\frac{1}{a}\int_0^\infty \cos x e^{-ax}dx$

substitute: $u = \cos x, dv = e^{-ax}dx \Rightarrow du = -\sin x dx, v = \frac{e^{-ax}}{-a}$

$\Rightarrow f'(a) = -\frac{1}{a}\left[\cos x\cdot\frac{e^{-ax}}{-a}\Big|_0^\infty - \int_0^\infty \sin x\frac{e^{-ax}}{a}dx\right] = -\frac{1}{a}\left[\frac{1}{a} + \frac{1}{a}f'(a)\right] \Rightarrow f'(a) = -\frac{1}{a}\left[\frac{1}{a} + \frac{1}{a}f'(a)\right] \Rightarrow f'(a)\left[1 + \frac{1}{a^2}\right] = -\frac{1}{a^2} \Rightarrow f'(a)\left[a^2+1\right] = -1 \Rightarrow f'(a) = -\frac{1}{1+a^2} \Rightarrow f(a) = -\tan^{-1}(a) + C$; to evaluate the constant note that $\lim_{a\to\infty} f(a) = \lim_{a\to\infty}\int_0^\infty \frac{\sin x}{x}e^{-ax}dx = \int_0^\infty \frac{\sin x}{x}\lim_{a\to\infty}[e^{-ax}]dx = \int_0^\infty \frac{\sin x}{x}\cdot 0\,dx = \int_0^\infty 0\,dx = 0$; hence we have $\lim_{a\to\infty} f(a) = \lim_{a\to\infty} -\tan^{-1}(a) + C = -\tan^{-1}(\infty) + C = -\frac{\pi}{2} + C = 0 \Rightarrow C = \frac{\pi}{2} \Rightarrow f(a) = -\tan^{-1}(a) + \frac{\pi}{2} \Rightarrow f(0) = -\tan^{-1}(0) + \frac{\pi}{2} = \frac{\pi}{2}$ ok

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Question 44 Solution

$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \cdots$; it is a convergent alternating series, so $|\cos x - 1| \leq \frac{1}{2}x^2$, and $\left|\cos x - (1 - \frac{1}{2}x^2)\right| \leq \frac{1}{4!}x^4$; setting $x = \frac{\pi}{5}$ gives the result

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Question 45 Solution

$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\int_0^x(1 - t^2 + \frac{t^4}{2} + ...)dt = \frac{2}{\sqrt{\pi}}(x - \frac{x^3}{3} + \frac{x^5}{10} + ...)$

Question 46 Solution

a) Use the geometric series, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

$\frac{a}{a+b} = \frac{a/b}{a/b+b/b} = \frac{a}{b}\cdot\frac{1}{1+\frac{a}{b}} = \frac{a}{b}\cdot\frac{1}{1-\left(-\frac{a}{b}\right)} = \frac{a}{b}\cdot\sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^n = \frac{a}{b}\left(1 - \frac{a}{b} + \frac{a^2}{b^2} + \cdots\right) = \frac{a}{b} - \frac{a^2}{b^2} + \frac{a^3}{b^3} + \cdots$

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b) Use the binomial series. $(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \cdots$ for $-1 < x < 1$.

$\frac{\sqrt{R^2-r^2}}{R} = \sqrt{1 - \frac{r^2}{R^2}} = \left(1 - \frac{r^2}{R^2}\right)^{1/2} = 1 + \frac{1}{2}(-\frac{r^2}{R^2}) + \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2}(-\frac{r^2}{R^2})^2 + \cdots = 1 - \frac{1}{2}\frac{r^2}{R^2} - \frac{1}{8}\frac{r^4}{R^4} + \cdots$

Question 47 Solution

Start from the Taylor series of $f(x)$ about $x = a$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots,$$

replace $x \to x + h, a \to x$. Then $x - a \to (x+h) - x = h$ and the Taylor series becomes

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \cdots.$$

Question 48 Solution

a) let $y = 0 \Rightarrow x = \pm(1+\epsilon)$, let $x = 0 \Rightarrow y = \pm 1$

b) Solve for y: $y = f(x) = \pm\sqrt{1 - \left(\frac{x}{1+\epsilon}\right)^2} \Rightarrow A(\epsilon) = 4\int_0^{1+\epsilon}\sqrt{1 - \left(\frac{x}{1+\epsilon}\right)^2}dx$

c) $A(\epsilon) = 4(1+\epsilon)\int_0^{1+\epsilon}\sqrt{1-\left(\frac{x}{1+\epsilon}\right)^2}d\frac{x}{1+\epsilon} = 4(1+\epsilon)\int_0^1\sqrt{1-u^2}du = 4(1+\epsilon)\frac{\pi}{4} = (1+\epsilon)\pi = \pi+\pi\epsilon$

<u>Question 49 Solution</u>

$V(x) = -\frac{Gm_1}{|x-x_1|} - \frac{Gm_2}{|x-x_2|}$ for $x\to\infty$, so we may assume $x > x_1, x_2$ and hence $V(x) = -\frac{Gm_1}{x-x_1} - \frac{Gm_2}{x-x_2}$

set $y = 1/x$, so $x = 1/y$, and expand the potential in powers of y

$V(1/y) = -\frac{Gm_1}{1/y-x_1} - \frac{Gm_2}{1/y-x_2} = -\frac{Gm_1y}{1-x_1y} - \frac{Gm_2y}{1-x_2y}$

by geometric series we have $\frac{1}{1-x_1y} = 1 + x_1y + (x_1y)^2 + \cdots$ and similarly for x_2

$V(1/y) = -Gm_1y(1+x_1y+(x_1y)^2+\cdots) - Gm_2y(1+x_2y+(x_2y)^2+\cdots)$

$= -G(m_1+m_2)y - G(m_1x_1+m_2x_2)y^2 - G(m_1x_2^2+m_2x_2^2)y^3 - \cdots$

change back to $x = 1/y$

$V(x) = -G(m_1+m_2)\frac{1}{x} - G(m_1x_1+m_2x_2)\frac{1}{x^2} - G(m_1x_2^2+m_2x_2^2)\frac{1}{x^3} - \cdots = \frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^3}-\cdots$

$\Rightarrow a = -G(m_1+m_2), b = -G(m_1x_1+m_2x_2), c = -G(m_1x_2^2+m_2x_2^2)$

<u>Question 50 Solution</u>

a) $\lim\limits_{r\to 0} V(r) = \lim\limits_{r\to 0} V_0((\frac{r_0}{r})^{12} - 2(\frac{r_0}{r})^6) = V_0 \lim\limits_{r\to 0}\frac{r_0^{12}-2r_0^6r^6}{r^{12}} = V_0\frac{r_0^{12}}{0^+} = \infty$

$\lim\limits_{r\to\infty} V(r) = \lim\limits_{r\to\infty} V_0((\frac{r_0}{r})^{12} - 2(\frac{r_0}{r})^6) = V_0 \lim\limits_{r\to\infty}\frac{r_0^{12}-2r_0^6r^6}{r^{12}} = V_0\cdot 0 = 0$

b) $V'(r) = V_0((-12)\frac{r_0^{12}}{r^{13}} - 2(-6)\frac{r_0^6}{r^7}) = V_0(\frac{-12r_0^{12}}{r^{13}} + \frac{12r_0^6}{r^7}) = V_0\frac{-12r_0^{12}+12r_0^6r^6}{r^{13}}$

$V'(r) = 0 \Leftrightarrow -12r_0^{12}+12r_0^6r^6 = 0 \Leftrightarrow 12r_0^6r^6 = 12r_0^{12} \Leftrightarrow r^6 = r_0^6 \Leftrightarrow r = r_0$.

d) Find the quadratic Taylor approximation at <u>$x = x_0$</u>, ie., $c_0 + c_1(x-x_0) + c_2(x-x_0)^2$ using the Theorem

$(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \cdots$ for $-1 < x < 1$

$V(x) = V_0\left[\left(\frac{x_0}{x}\right)^{12} - 2\left(\frac{x_0}{x}\right)^6\right] = V_0\left[\left(\frac{x}{x_0}\right)^{-12} - 2\left(\frac{x}{x_0}\right)^{-6}\right] = V_0\left[\left(\frac{x-x_0+x_0}{x_0}\right)^{-12} - 2\left(\frac{x-x_0+x_0}{x_0}\right)^{-6}\right]$

$= V_0\left[\left(1+\frac{x-x_0}{x_0}\right)^{-12} - 2\left(1+\frac{x-x_0}{x_0}\right)^{-6}\right]$

using the above Theorem

$\left(1+\frac{x-x_0}{x_0}\right)^{-12} = 1 - 12\frac{x-x_0}{x_0} + \frac{(-12)\cdot(-12-1)}{2}\left(\frac{x-x_0}{x}\right)^2 + \cdots = 1 - \frac{12}{x_0}(x-x_0) + \frac{78}{x_0^2}(x-x_0)^2 + \cdots$

$\left(1+\frac{x-x_0}{x_0}\right)^{-6} = 1 - 6\frac{x-x_0}{x_0} + \frac{(-6)\cdot(-6-1)}{2}\left(\frac{x-x_0}{x}\right)^2 + \cdots = 1 - \frac{6}{x_0}(x-x_0) + \frac{21}{x_0^2}(x-x_0)^2 + \cdots$

$V(x) = V_0\left[\left(1 - \frac{12}{x_0}(x-x_0) + \frac{78}{x_0^2}(x-x_0)^2 + \cdots\right) - 2\left(1 - \frac{6}{x_0}(x-x_0) + \frac{21}{x_0^2}(x-x_0)^2\right)\right] + \cdots$

$= V_0\left[-1 + \frac{36}{x_0^2}(x-x_0)^2\right] + \cdots = -V_0 + 36\frac{V_0}{x_0^2}(x-x_0)^2 + \cdots$

$T_2(x) = -V_0 + 36\frac{V_0}{x_0^2}(x-x_0)^2$

e) We know that work equals the integral of the force:

$W = \int\limits_{r_0}^{\infty} f(r)dr = \int\limits_{r_0}^{\infty} -V'(r)dr = -V(r)|_{r_0}^{\infty} = -0 - (-V(r_0)) = V(r_0) = V_0(1-2) = -V_0$

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binomial series

<u>Question 51 Solution</u>

Using the Binomial Series Theorem,

$(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \cdots$ for $-1 < x < 1$

$(1+x^2)^k = 1 + kx^2 + \frac{k(k-1)}{2}x^4 + \frac{k(k-1)(k-2)}{3!}x^6 + \frac{k(k-1)(k-2)(k-3)}{4!}x^8 + \cdots$

$(1+x^2)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^4 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^6 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}x^8 + \cdots = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \frac{5}{128}x^8 + \cdots$

This is an alternating series and the assumptions in the AST apply.

Using the second order Taylor approximation, $T_2(x) = 1 + \frac{1}{2}x^2$, we have $|s - T_2| < \frac{1}{8}x^4$, where s is the exact value.
$\int_0^1 \sqrt{1+x^2}dx \approx \int_0^1 (1 + \frac{1}{2}x^2)dx = x + \frac{1}{6}x^3\Big|_0^1 = \frac{7}{6}$ and the error is less than $\int \frac{1}{8}x^4 dx = \frac{1}{40}x^5\Big|_0^1 = \frac{1}{40}$

<u>Question 52 Solution</u>
Consider the expansion $\dfrac{1}{\sqrt{1-2ax+x^2}} = c_0 + c_1x + c_2x^2 + \cdots$, where a is a constant. Find c_0, c_1, c_2 in terms of a. Apply the binomial expansion.

$$\frac{1}{\sqrt{1-2ax+x^2}} = (1-2ax+x^2)^{-1/2} = (1+(-2ax+x^2))^{-1/2}$$
$$= 1 + (-\tfrac{1}{2})(-2ax+x^2) + \tfrac{(-\frac{1}{2})(-\frac{3}{2})}{2}(-2ax+x^2)^2 + \cdots = 1 + ax - \tfrac{1}{2}x^2 + \tfrac{3}{8}(4a^2x^2 - 4ax^3 + x^4) + \cdots$$
$$= 1 + ax + (-\tfrac{1}{2} + \tfrac{3}{2}a^2)x^2 + \cdots \Rightarrow c_0 = 1, c_1 = a, c_2 = -\tfrac{1}{2} + \tfrac{3}{2}a^2$$

<u>Question 53 Solution</u>
a) Show that $\binom{k+1}{n+1} = \binom{k}{n} + \binom{k}{n+1}$.
The left hand side is number of ways of choosing $n+1$ objects from a set of $k+1$ objects (disregarding the order in which the objects are chosen).
To understand the right hand side, assume all $k+1$ objects are white, randomly pick one object, color it red, then put it back. Now choose $n+1$ objects from these $k+1$ objects; there are two possibilities, if the red one is chosen, the number of ways is $\binom{k}{n}$ (it is equivalent to choosing n from k objects); if the red one is not chosen, the number of ways is $\binom{k}{n+1}$ (it is equivalent to choosing $n+1$ objects from k objects). Hence the left hand side equals the right hand side, since it is the same thing, choosing $n+1$ objects from $k+1$ objects.
$\binom{k+1}{n+1} = \frac{(k+1)!}{(n+1)!(k-n)!} = \frac{k!(k+1)}{(n+1)!(k-n)!} = \frac{k!(k-n)+k!(n+1)}{(n+1)!(k-n)!} = \frac{k!}{(n+1)!(k-n-1)!} + \frac{k!}{n!(k-n)!} = \binom{k}{n+1} + \binom{k}{n}$

b)
$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}
\end{array}
\qquad
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1
\end{array}$$

Denote elements in the triangle by $a_{k,n}$, nth element on kth row. Each row is obtained by adding the two entries diagonally above, $a_{k+1,n+1} = a_{k,n} + a_{k,n+1}$ which is equivalent to $\binom{k+1}{n+1} = \binom{k}{n} + \binom{k}{n+1}$.
c) $(a+b)^6 = \binom{6}{0}a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}ab^5 + \binom{6}{6}b^6$
$\quad = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

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complex numbers
<u>Question 54 Solution</u>
a) $1 + i = \sqrt{2}e^{\frac{\pi}{4}i}$ $\quad$ ($x = 1, y = 1$ already in Cartesian form)
b) $(1+i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i = 2e^{\frac{\pi}{2}i}$
c) $(1+i)^3 = (1+i)^2(1+i) = 2i(1+i) = -2 + 2i = 2\sqrt{2}e^{\frac{3\pi}{4}i}$
d) $\frac{1}{1+i} = \frac{1\cdot(1-i)}{(1+i)\cdot(1-i)} = \frac{1-i}{1-i^2} = \frac{1-i}{1-(-1)} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i = \frac{1}{\sqrt{2}}e^{\frac{-\pi}{4}i}$
e) $\sqrt{1+i} = \left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^{\frac{1}{2}} = 2^{\frac{1}{4}}e^{\frac{\pi}{8}i} = 2^{\frac{1}{4}}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right) = 2^{\frac{1}{4}}\cos\frac{\pi}{8} + i\,2^{\frac{1}{4}}\sin\frac{\pi}{8}$

<u>Question 55 Solution</u>
a) binomial expansion
$(1+i)^6 = 1 + 6i + 15i^2 + 20i^3 + 15i^4 + 6i^5 + i^6 = 1 + 6i - 15 - 20i + 15 + 6i - 1 = -8i$

b) polar form
$1+i=\sqrt{2}e^{\pi i/4} \Rightarrow (1+i)^6=(\sqrt{2}e^{\pi i/4})^6=2^3e^{3\pi i/2}=8(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2})=-8i$

Question 56 Solution

a) $z^2+2z-2=0 \Rightarrow z_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{4+8}}{2}=-1\pm\sqrt{3}$

b) $z^2+2z+2=0 \Rightarrow z_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-2\pm\sqrt{4-8}}{2}=\frac{-2\pm\sqrt{-4}}{2}=\frac{-2\pm2\sqrt{-1}}{2}=-1\pm i$

c) $z^2=(re^{i\theta})^2=r^2e^{2i\theta}=1 \Rightarrow r=1, 2\theta=0,2\pi \Rightarrow \theta=0,\pi \Rightarrow z_{1,2}=e^0,e^{\pi i}=\pm1$

d) $z^3=r^3e^{3i\theta}=-1 \Rightarrow r=1, 3\theta=\pm\pi,3\pi \Rightarrow \theta=\pm\frac{\pi}{3},\pi \Rightarrow z_{1,2,3}=e^{\pi i},e^{\pm\pi i/3}=-1,\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

e) $z^4=1 \Rightarrow z^2=\pm1 \Rightarrow z_{1,2,3,4}=\pm1,\pm i$

f) $e^z=1$: on the real axis there is one root $z=0$, but in the complex plane there are infinitely many roots, let $z=x+iy$, then $e^z=e^{x+iy}=e^x(\cos y+i\sin y)=1 \Rightarrow x=0, y=2\pi n$, where n is any integer, roots are $z_n=2\pi in, n=\pm1,\pm2,\pm3,\ldots$

Question 57 Solution

$(\cos\theta+i\,\sin\theta)^n=\left(e^{i\theta}\right)^n=e^{in\theta}=\cos n\theta+i\sin n\theta$

Question 58 Solution

$e^{i(a+b)}=e^{ia}\cdot e^{ib} \Rightarrow \cos(a+b)+i\sin(a+b)=(\cos a+i\sin a)\cdot(\cos b+i\sin b)=\cos a\cos b-\sin a\sin b+i(\sin a\cos b+\cos a\sin b)$

take real and imaginary parts : $\cos(a+b)=\cos a\,\cos b-\sin a\,\sin b$, $\sin(a+b)=\sin a\,\cos b+\cos a\,\sin b$

Question 59 Solution

a1) $\int e^{ax}\cos bx\,dx=\int e^{ax}\frac{1}{b}\,d\sin bx=\frac{1}{b}e^{ax}\cdot\sin bx-\int\frac{1}{b}\sin bx\,de^{ax}=\frac{1}{b}e^{ax}\cdot\sin bx-\int\frac{a}{b}e^{ax}\sin bx\,dx=\frac{1}{b}e^{ax}\cdot\sin bx-\int\frac{a}{b}e^{ax}\left(-\frac{1}{b}\right)d\cos bx=\frac{1}{b}e^{ax}\cdot\sin bx+\int\frac{a}{b^2}e^{ax}\,d\cos bx=\frac{1}{b}e^{ax}\cdot\sin bx+\frac{a}{b^2}e^{ax}\cdot\cos bx-\frac{a}{b^2}\int\cos bx\,de^{ax}=\frac{1}{b}e^{ax}\cdot\sin bx+\frac{a}{b^2}e^{ax}\cdot\cos bx-\frac{a^2}{b^2}\int e^{ax}\cos bx\,dx \Rightarrow \underline{\int e^{ax}\cos bx\,dx}=\frac{1}{b}e^{ax}\cdot\sin bx+\frac{a}{b^2}e^{ax}\cdot\cos bx-\frac{a^2}{b^2}\underline{\int e^{ax}\cos bx\,dx} \Rightarrow \left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cos bx\,dx=\frac{1}{b}e^{ax}\cdot\sin bx+\frac{a}{b^2}e^{ax}\cdot\cos bx \Rightarrow \int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}\left(b\cdot\sin bx+a\cdot\cos bx\right)$

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a2) $\int e^{ax}\sin bx\,dx=\int e^{ax}\left(-\frac{1}{b}\right)d\cos bx=-\frac{1}{b}e^{ax}\cdot\cos bx+\int\frac{1}{b}\cos bx\,de^{ax}=-\frac{1}{b}e^{ax}\cdot\cos bx+\int\frac{a}{b}e^{ax}\cos bx\,dx=-\frac{1}{b}e^{ax}\cdot\cos bx+\int\frac{a}{b}e^{ax}\cdot\frac{1}{b}\,d\sin bx=-\frac{1}{b}e^{ax}\cdot\cos bx+\int\frac{a}{b^2}e^{ax}\,d\sin bx=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b^2}e^{ax}\cdot\sin bx-\frac{a}{b^2}\int\sin bx\,de^{ax}=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b^2}e^{ax}\cdot\sin bx-\frac{a^2}{b^2}\int e^{ax}\sin bx\,dx \Rightarrow \underline{\int e^{ax}\sin bx\,dx}=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b^2}e^{ax}\cdot\sin bx-\frac{a^2}{b^2}\underline{\int e^{ax}\sin bx\,dx} \Rightarrow \left(1+\frac{a^2}{b^2}\right)\int e^{ax}\sin bx\,dx=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b^2}e^{ax}\cdot\sin bx \Rightarrow \int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left(a\cdot\sin bx-b\cdot\cos bx\right)$

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b) $e^{(a+ib)x}=e^{ax+ibx}=e^{ax}\cdot e^{ibx}=e^{ax}\left(\cos bx+i\,\sin bx\right)$

$\int e^{(a+ib)x}dx=\frac{1}{a+ib}e^{(a+ib)x}=\frac{1\cdot(a-ib)}{(a+ib)\cdot(a-ib)}e^{(a+ib)x}=\frac{a-ib}{a^2+b^2}e^{(a+ib)x}$

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c) $\int e^{(a+ib)x}dx=\int\left(e^{ax}\cos bx+i\,e^{ax}\sin bx\right)dx=\int e^{ax}\cos bx\,dx+i\int e^{ax}\sin bx\,dx=\frac{a-ib}{a^2+b^2}e^{(a+ib)x}$
$\frac{a-ib}{a^2+b^2}e^{(a+ib)x}=\frac{a-ib}{a^2+b^2}\left(e^{ax}\cos bx+i\,e^{ax}\sin bx\right)=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+i\left[\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\sin bx)\right]$
$\Rightarrow \int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)$ and $\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\sin bx)$

Question 60 Solution

a) $e^{ix}=\cos x+i\,\sin x \Rightarrow e^{-ix}=\cos x-i\,\sin x \Rightarrow e^{ix}+e^{-ix}=2\cos x \Rightarrow \cos x=\frac{e^{ix}+e^{-ix}}{2}$

b) $e^{ix}=\cos x+i\,\sin x \Rightarrow e^{-ix}=\cos x-i\,\sin x \Rightarrow e^{ix}-e^{-ix}=2i\sin x \Rightarrow \sin x=\frac{e^{ix}-e^{-ix}}{2i}$

c) $\frac{d}{dx}\cos x=\frac{d}{dx}\left(\frac{e^{ix}+e^{-ix}}{2}\right)=\frac{ie^{ix}-ie^{-ix}}{2}=\frac{i(e^{ix}-e^{-ix})}{2}=\frac{i^2(e^{ix}-e^{-ix})}{2i}=\frac{-(e^{ix}-e^{-ix})}{2i}=-\sin x$

d) $\frac{d}{dx}\sin x = \frac{d}{dx}(\frac{e^{ix}-e^{-ix}}{2i}) = \frac{ie^{ix}+ie^{-ix}}{2i} = \frac{e^{ix}+e^{-ix}}{2} = \cos x$

e) $\cos^2 x+\sin^2 x = (\frac{e^{ix}+e^{-ix}}{2})^2+(\frac{e^{ix}-e^{-ix}}{2i})^2 = \frac{e^{2ix}+2+e^{-2ix}}{4}+\frac{e^{2ix}-2+e^{-2ix}}{-4} = \frac{e^{2ix}+2+e^{-2ix}}{4}-\frac{e^{2ix}-2+e^{-2ix}}{4} = 1$

f) $\cos^2 x-\sin^2 x = (\frac{e^{ix}+e^{-ix}}{2})^2-(\frac{e^{ix}-e^{-ix}}{2i})^2 = (\frac{e^{2ix}+2+e^{-2ix}}{4})-(\frac{e^{2ix}-2+e^{-2ix}}{-4}) = \frac{e^{2ix}+e^{-2ix}}{2} = \cos 2x$

g) $\sin 2x = \frac{e^{2ix}-e^{-2ix}}{2i} = \frac{(e^{ix}-e^{-ix})(e^{ix}+e^{-ix})}{2i} = 2\frac{(e^{ix}-e^{-ix})}{2i}\cdot\frac{(e^{ix}+e^{-ix})}{2} = 2\sin x\cos x$

h) $\cos^2 x = (\frac{e^{ix}+e^{-ix}}{2})^2 = \frac{e^{2ix}+2+e^{-2ix}}{4} = \frac{1}{2}+\frac{1}{2}(\frac{e^{2ix}+e^{-2ix}}{2}) = \frac{1}{2}+\frac{1}{2}\cos 2x$

i) $\cos^3 x = (\frac{e^{ix}+e^{-ix}}{2})^3 = \frac{e^{3ix}+3e^{ix}+3e^{-ix}+e^{-3x}}{8} = \frac{1}{4}(\frac{e^{3ix}+e^{-3ix}}{2})+\frac{3}{4}(\frac{e^{ix}+e^{-ix}}{2}) = \frac{3}{4}\cos x+\frac{1}{4}\cos 3x$