These are sample problems of the type that may appear on the exam - students should work on these problems, but they should not be turned in for grading. The exam will have around 5-6 problems, much less than the review sheet.

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1. True or False. Justify your answer with a reason or counterexample.

a) $\displaystyle\sum_{i=1}^{12} 2i = 156$ b) $\displaystyle\sum_{i=1}^{12}\left(\frac{1}{i}-\frac{1}{i+1}\right) = \frac{12}{13}$ c) $\displaystyle\sum_{i=0}^{n}(n-i)^2 = \sum_{i=0}^{n} i^2$ d) $\displaystyle\left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^3$

e) $1+3+5+7+\cdots+(2n-1) = n^2$ f) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{512} < 2$ g) $\displaystyle\sum_{i=1}^{9}((i+1)^3-i^3)) = 999$

h) If an integral $\int_a^b f(x)\,dx$ is computed using the right-hand Riemann sum and the number of intervals n is doubled, then the error is approximately also doubled.

i) $\displaystyle\int e^{x^2}dx = \frac{e^{x^2}}{2x}$ j) $\displaystyle\int xe^{x^2}dx = \frac{1}{2}e^{x^2}$ k) $\displaystyle\int_0^\infty e^{-x}\cos x\;dx = \int_0^\infty e^{-x}\sin x\;dx$

l) A spring has natural length 10 cm. If 2 J of work are needed to stretch the spring from length 10 cm to 15 cm, then 8 J of work are needed to stretch it from length 10 cm to 20 cm.

m) A cable hanging from the top of a tall building has length L m, uniform cross-sectional area A m^2, and density ρ kg/m^3. The cable is pulled to the top of the building. If the length of the cable is doubled, then the work is also doubled.

n) The area under the graph of $y = \dfrac{1}{x^2}$ from $x = 1$ to $x = \infty$ is finite.

o) If $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to\infty} g(x) = 0$, then $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} = 1$.

p) If $0 \le f(x) \le g(x)$ for $x \ge 1$ and $\int_1^\infty g(x)dx$ converges, then $\displaystyle\int_1^\infty f(x)dx$ also converges.

q) The error function, defined by $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}dt$, satisfies $\mathrm{erf}''(0) = \mathrm{erf}(0)$.

r) $\displaystyle\int_0^{\pi/2}\sin^2 x dx = \int_0^{\pi/2}\cos^2 x dx$ (hint : this is true; try to prove it using 3 different methods)

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section 1.2 area, 1.3 definite integral, 1.4 FTC

2. Express the integral as a limit of Riemann sums, evaluate the limit, check by the FTC.

a) $\displaystyle\int_0^2 xdx$ b) $\displaystyle\int_0^1 x^3dx$ c) $\displaystyle\int_a^b xdx$ d) $\displaystyle\int_a^b x^2dx$ e) $\displaystyle\int_0^1 e^{-x}dx$

3. Evaluate the limit by any means.

a) $\lim\limits_{x\to\infty} xe^{-x}$ b) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3\frac{1}{n}$ c) $\displaystyle\lim_{x\to 0}\frac{1}{x}\int_0^x f(t)\,dt$ d) $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x}$ e) $\displaystyle\lim_{r\to 1}\frac{1-r^{10}}{1-r}$

4. Find the antiderivative.

a) $\displaystyle\int xe^{-x^2}dx$ b) $\displaystyle\int x^2e^{-x}dx$ c) $\displaystyle\int x\sin x dx$ d) $\displaystyle\int\frac{dx}{4-x^2}$ e) $\displaystyle\int\frac{dx}{\sqrt{4-x^2}}$ f) $\displaystyle\int\sqrt{4-x^2}\,dx$

5. Prove. a) $\displaystyle\frac{1}{20} \le \int_0^1\frac{x^9}{1+x}dx \le \frac{1}{10}$ b) $\displaystyle\int_0^1 x\,(1-x)^{11}dx = \frac{1}{156}$

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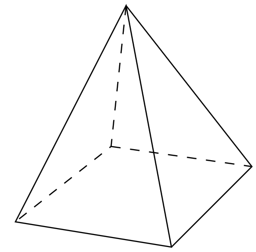
section 1.5 work

6. A force of 30 N is needed to stretch a spring from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

7. Two ions each have charge q and they repel each other with force $f(r) = -\dfrac{q^2}{4\pi\epsilon_0\,r^2}$, where ϵ_0 is the vacuum permittivity, r is the distance between the ions, and the negative sign indicates that the force is repulsive. (a) If one ion is held fixed at $x = 0$, find the work W done in moving the other ion from $x = 3$ to $x = 2$. (b) Find W if these are sodium ions with charge $q = 1.5\cdot 10^{-19}$ C (coulomb), distance is measured in angstroms (Å$= 10^{-10}$ m), and $(4\pi\epsilon_0)^{-1} = 9\cdot 10^9\ \mathrm{N}\cdot\mathrm{m}^2/\mathrm{C}^2$.

8. A pyramid is built of stone with density $\rho\,\mathrm{kg/m^3}$. The base of the pyramid is a square, and the vertex is directly above the center of the base. The length of a side of the base is $L\,\mathrm{m}$ and the height of the vertex above the base is $H\,\mathrm{m}$. a) Derive a formula for the work done in building the pyramid (i.e. raising the stone from ground level to its level in the pyramid). b) If the length L and height H are doubled, by what factor does the work increase? c) Which requires more work, building the lower half or the upper half of the pyramid?



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section 1.6 improper integrals

9. Determine whether the integral converges or diverges. If it converges, find the value. If it diverges, give a reason.

a) $\int_1^\infty \frac{dx}{x^4}$ b) $\int_0^\infty x^2 e^{-x} dx$ c) $\int_0^\infty e^{-x}\sin x dx$ d) $\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$ e) $\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$

f) $\int_{-r}^{r} \frac{dx}{\sqrt{r^2 - x^2}}$ g) $\int_1^\infty \frac{dx}{1+x^2}$ h) $\int_1^\infty \frac{dx}{\sqrt{1+x^2}}$ i) $\int_1^\infty \frac{x}{\sqrt{1+x^2}}\, dx$ j) $\int_1^\infty \frac{dx}{x^2-1}$

k) $\int_0^1 \frac{dx}{\sqrt{x}}$ l) $\int_0^1 \frac{dx}{x^{3/2}}$ m) $\int_0^1 \frac{dx}{1-x}$ n) $\int_0^\infty \frac{\ln x}{1+x^2}\, dx$ (hint: substitute $u = x^{-1}$)

10. A patient receives an intravenous drug at the rate $r(t) = 2te^{-2t}$ ml/sec, where t is the time in seconds since the treatment started. (a) Find the total dose the patient receives in the limit $t \to \infty$. (b) What fraction of the total dose is received in the first 5 seconds?

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miscellaneous

11. a) Show that $\frac{x}{x^2+1} \sim \frac{1}{x}$ as $x \to \infty$. b) Sketch the graphs of $y_1 = \frac{x}{x^2+1}$ and $y_2 = \frac{1}{x}$ for $x \geq 0$ on the same plot. Label each curve. Do the curves intersect?

12. A tank contains radioactive waste from a nuclear reactor. Due to radioactive decay, a fraction a of the waste in the tank at the start of a year remains in the tank at the end of the year, where $0 < a < 1$, and an additional b kg of waste is added to the tank each year. Let w_0 kg be the amount of waste in the tank on January 1, 2000, and let w_n kg be the amount of waste in the tank after n years have passed. a) Write the expression for w_n in terms of w_{n-1}. b) Find w_1, w_2, w_3, w_4. c) Find the formula for w_n in terms of a, b, n, w_0. d) Let $w_0 = 50$ kg, $a = 0.9$, and $b = 10$ kg. The capacity of the tank is 100 kg; will the waste ever exceed that value?

13. Define the <u>average value</u> of a function $f(x)$ on the interval $[a, b]$ by $f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)dx$ and define the <u>root mean square value</u> by $f_{\text{rms}} = \left(\frac{1}{b-a}\int_a^b (f(x))^2 dx\right)^{1/2}$.

Find $f_{\text{avg}}, f_{\text{rms}}$ for (a) $f(x) = \sin x$, (b) $f(x) = \sin 2x$, (c) $f(x) = \sin^2 x$ on the interval $[0, 2\pi]$.

hint : compute f_{rms}^2, then take square root, use $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x, \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

14. The current and voltage in an electric AC circuit are $I(t) = I_0 \cos\omega t, V(t) = V_0\cos(\omega t + \phi)$, where I_0 is peak current, V_0 is peak voltage, t is time, ω is angular frequency, and ϕ is the phase shift. The work done by an electric AC circuit in one cycle is $W = \int_0^T P(t)dt$, where $T = 2\pi/\omega$ is the period and $P(t) = I(t)\cdot V(t)$ is the power (the rate at which the circuit does work).

a) Find the work done over 1 cycle when the current and voltage are in phase ($\phi = 0$).

b) Find the work done over 1 cycle when the current and voltage are 90° out of phase ($\phi = \pi/2$). You may use the formulas $\cos(a+b) = \cos a\cos b - \sin a \sin b, \sin 2x = 2\sin x\cos x$.

15. Sketch the graph of the function on the given interval.

a) $e^x, (-\infty, \infty)$ b) $\ln x, (0, \infty)$ c) $\tan x, [-2\pi, 2\pi]$ d) $\tan^{-1} x, (-\infty, \infty)$ e) $\sin^2 x, [0, 2\pi]$