

Math 156 Midterm Exam 1 Review Solutions Fall 2023

1a) True $\sum_{i=1}^n i = \frac{n(n+1)}{2} \Rightarrow \sum_{i=1}^{12} (2i) = 2 \sum_{i=1}^{12} i = 2 \frac{12 \cdot 13}{2} = 156$ (can also be done by direct computation)

1b) True $\sum_{i=1}^{12} \left(\frac{1}{i} - \frac{1}{i+1} \right) = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{12} - \frac{1}{13} \right) = 1 - \frac{1}{13} = \frac{12}{13}$

1c) True The two sums contain the same terms in the opposite order. (This could be justified with the substitution $j = n - i$.) Alternatively, using our known formula for $\sum_{i=0}^n i$, we have

$$\sum_{i=0}^n (n-i)^2 = \sum_{i=0}^n (n^2 - 2in + i^2) = n^2 \sum_{i=0}^n 1 - 2n \sum_{i=0}^n i + \sum_{i=0}^n i^2 = n^2(n+1) - 2n \frac{n(n+1)}{2} + \sum_{i=0}^n i^2 = \sum_{i=0}^n i^2.$$

1d) True In class and on homework we derived $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$.

Then $\left(\sum_{i=1}^n i \right)^2 = \left(\frac{n(n+1)}{2} \right)^2 = \sum_{i=1}^n i^3$.

1e) True $1 + 3 + 5 + 7 + \cdots + (2n-1) = \sum_{i=1}^n (2i-1) = 2 \sum_{i=1}^n i - \sum_{i=1}^n 1 = 2 \cdot \frac{n(n+1)}{2} - n = n^2$

1f) True $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{512} = \sum_{i=0}^9 \left(\frac{1}{2} \right)^i = \frac{1 - \left(\frac{1}{2} \right)^{10}}{1 - \frac{1}{2}} = \frac{1 - \frac{1}{1024}}{\frac{1}{2}} = 2 \left(1 - \frac{1}{1024} \right) < 2 \cdot 1 = 2$

1g) False If the number of intervals using any Riemann sum increases, then the error will decrease, not increase. In the case of the right-hand Riemann sum, if the number of intervals is doubled, the error is approximately cut in half.

1h) False $\frac{d}{dx} \left(\frac{e^{x^2}}{2x} \right) = \frac{2x \cdot 2xe^{x^2} - e^{x^2} \cdot 2}{x^4} = e^{x^2} \left(\frac{4x^2 - 2}{x^4} \right) \neq e^{x^2}$

1i) False By the FTC and chain rule, $\frac{d}{dx} \int_0^{x^3} \sqrt{1+t^2} dt = \sqrt{1+x^6} \cdot 3x^2$.

1j) True Apply integration by parts, $\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$, to the first integral.

$$u = e^{-x}, dv = \cos x dx \Rightarrow du = -e^{-x} dx, v = \sin x$$

$$\int_0^\infty e^{-x} \cos x dx = e^{-x} \sin x \Big|_0^\infty + \int_0^\infty e^{-x} \sin x dx = 0 - 0 + \int_0^\infty e^{-x} \sin x dx = \int_0^\infty e^{-x} \sin x dx$$

1k) True The work needed to stretch the spring from 10 cm to 15 cm is $\int_0^5 kx dx = \frac{25}{2}k = 2$ J, so $k = \frac{4}{25}$. Then the work needed to stretch the spring from 10 cm to 20 cm is $\int_0^{10} \frac{4}{25}x dx = \frac{4}{25} \cdot 50 = 8$ J.

1l) False Let x be a vertical coordinate with $x = 0$ at the top of the building and $x = L$ at the bottom of the cable. Consider a slice of the cable at position x_i . The volume of the slice is $A\Delta x$; the mass of the slice is $\rho A\Delta x$; the force acting on the slice is $\rho g A\Delta x$; the work done in pulling the slice to the top of the building is $\rho g A\Delta x \cdot x_i$. Hence the total work is $W = \lim_{n \rightarrow \infty} \rho g A x_i \Delta x = \int_0^L \rho g A x dx = \frac{1}{2} \rho g A L^2$. If the length of the cable is doubled from L to $2L$, then the work becomes $W = \frac{1}{2} \rho g A (2L)^2 = 2 \rho g A L^2$, which is four times the work done in raising a cable of length L , not double.

1m) True $\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x} \Big|_1^\infty = 0 - (-1) = 1$; the result also follows from the p -test with $p = 2$

1n) False Here's a counterexample. Let $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{2x}$, then $\lim_{x \rightarrow \infty} f(x) = 0$ and $\lim_{x \rightarrow \infty} g(x) = 0$, but

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2x}} = \lim_{x \rightarrow \infty} 2 = 2. \text{ Of course there are many other valid counterexamples.}$$

1o) True This follows from the comparison test.

1p) True Note that $\operatorname{erf}(0) = 0$. Then by the FTC we have $\operatorname{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$, so $\operatorname{erf}''(x) = \frac{2}{\sqrt{\pi}} (-2x) e^{-x^2}$, and hence $\operatorname{erf}''(0) = 0$.

1q) True

method 1: sketch the graph of each function, argue that the area is the same by symmetry

method 2: use integration by parts, $u = \sin x, dv = \sin x \Rightarrow du = \cos x, v = -\cos x \Rightarrow \int_0^{\pi/2} \sin^2 x dx = \sin x \cdot -\cos x \Big|_{x=0}^{\pi/2} - \int_0^{\pi/2} -\cos^2 x dx = \int_0^{\pi/2} \cos^2 x dx$

method 3: use trigonometric identities, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

2. Express the integral as a limit of Riemann sums, evaluate the limit, and check by the FTC.

2a. $\int_0^2 x dx \Rightarrow a = 0, b = 2, \Delta x = \frac{b-a}{n} = \frac{2}{n}, x_i = a + i\Delta x = \frac{2i}{n}, f(x) = x, f(x_i) = x_i = \frac{2i}{n}$

Riemann sums: $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \frac{2i}{n} = \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n i = \lim_{n \rightarrow \infty} \frac{4}{n^2} \frac{n(n+1)}{2} = 2$

FTC: $\int_0^2 x dx = \frac{x^2}{2} \Big|_0^2 = 2 - 0 = 2$

2b. $\int_0^1 x^3 dx \Rightarrow a = 0, b = 1, \Delta x = \frac{b-a}{n} = \frac{1}{n}, x_i = a + i\Delta x = \frac{i}{n}, f(x) = x^3, f(x_i) = x_i^3 = \left(\frac{i}{n}\right)^3$

Riemann sums: $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(\frac{i}{n}\right)^3 = \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{i=1}^n i^3 = \lim_{n \rightarrow \infty} \frac{1}{n^4} \left[\frac{n(n+1)}{2}\right]^2 = 1/4$

FTC: $\int_0^1 x^3 dx = \frac{x^4}{4} \Big|_0^1 = 1/4 - 0 = 1/4$

2c. $\int_a^b x dx \Rightarrow \Delta x = \frac{b-a}{n}, x_i = a + i\Delta x, f(x) = x, f(x_i) = x_i$

Riemann sums: $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n (a + i\Delta x) \Delta x$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(a + i \frac{b-a}{n} \right) \frac{b-a}{n}$$

$$= \lim_{n \rightarrow \infty} \left(a \frac{b-a}{n} \sum_{i=1}^n 1 + \frac{(b-a)^2}{n^2} \sum_{i=1}^n i \right)$$

$$= \lim_{n \rightarrow \infty} \left(a \frac{b-a}{n} n + \frac{(b-a)^2}{n^2} \frac{n(n+1)}{2} \right)$$

$$= a(b-a) + \frac{(b-a)^2}{2} = (b-a) \left(a + \frac{b-a}{2} \right) = (b-a) \left(\frac{b+a}{2} \right) = \frac{1}{2}(b^2 - a^2)$$

FTC: $\int_a^b x dx = \frac{x^2}{2} \Big|_a^b = \frac{b^2}{2} - \frac{a^2}{2}$

2d. $\int_a^b x^2 dx \Rightarrow \Delta x = \frac{b-a}{n}, x_i = a + i\Delta x, f(x) = x^2, f(x_i) = x_i^2$

Riemann sums: $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n x_i^2 \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n (a + i\Delta x)^2 \Delta x$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(a^2 + 2ia \frac{b-a}{n} + i^2 \frac{(b-a)^2}{n^2} \right) \frac{b-a}{n}$$

$$= \lim_{n \rightarrow \infty} \left(a^2 \frac{b-a}{n} \sum_{i=1}^n 1 + \frac{(b-a)^2}{n^2} 2a \sum_{i=1}^n i + \frac{(b-a)^3}{n^3} \sum_{i=1}^n i^2 \right)$$

$$= \lim_{n \rightarrow \infty} \left(a^2 \frac{b-a}{n} n + \frac{(b-a)^2}{n^2} 2a \frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3} \frac{n(n+1)(2n+1)}{6} \right)$$

$$= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3} = (b-a) \left(a^2 + a(b-a) + \frac{1}{3}(b-a)^2 \right)$$

$$= (b-a) \left(ab + \frac{1}{3}(b^2 - 2ab + a^2) \right) = \frac{1}{3}(b-a)(b^2 + ab + a^2) = \frac{1}{3}(b^3 - a^3)$$

FTC: $\int_a^b x^2 dx = \frac{x^3}{3} \Big|_a^b = \frac{b^3}{3} - \frac{a^3}{3}$

2e. $\int_0^1 e^{-x} dx \Rightarrow a = 0, b = 1, \Delta x = \frac{b-a}{n} = \frac{1}{n}, x_i = a + i\Delta x = \frac{i}{n}, f(x) = e^{-x}, f(x_i) = e^{-x_i} = e^{-i/n}$

$$\begin{aligned}
\text{Riemann sums : } \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n e^{-i/n} = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1 - (e^{-1/n})^{n+1}}{1 - e^{-1/n}} - 1 \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1 - e^{-(n+1)/n}}{1 - e^{-1/n}} \right) \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1 - e^{-1-1/n}}{1 - e^{-1/n}} \right) = \lim_{n \rightarrow \infty} (1 - e^{-1-1/n}) \cdot \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{1}{1 - e^{-1/n}} \right) \\
&= (1 - e^{-1}) \cdot \lim_{t \rightarrow 0} \frac{t}{1 - e^{-t}} = (1 - e^{-1}) \cdot \frac{0}{0} = (1 - e^{-1}) \cdot \lim_{t \rightarrow 0} \frac{1}{e^{-t}} = 1 - e^{-1}
\end{aligned}$$

note: l'Hôpital's rule is used in the last step; this derivation uses the right-hand Riemann sum; the left-hand Riemann sum can also be used, but you should get the same answer.

$$\text{FTC: } \int_0^1 e^{-x} dx = -e^{-x} \Big|_0^1 = -e^{-1} - (-1) = 1 - e^{-1}$$

3a. $\lim_{x \rightarrow \infty} x e^{-x} = 0 \cdot \infty = \lim_{x \rightarrow \infty} \frac{x}{e^x} = \frac{\infty}{\infty} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$; we used l'Hôpital's rule

3b. The sum is a Riemann sum for $f(x) = x^3$ on $1 \leq x \leq 2$. The limit is $\int_1^2 x^3 dx = [x^4/4]_1^2 = 15/4$.

3c. l'Hôpital's rule : $\lim_{x \rightarrow 0} \frac{\int_0^x f(t) dt}{x} = \lim_{x \rightarrow 0} \frac{f(x)}{1} = f(0)$; we assume f is continuous at 0

3d. l'Hôpital's rule : $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} \frac{e^x}{1} = 1$

3e. l'Hôpital's rule : $\lim_{r \rightarrow 1} \frac{1-r^{10}}{1-r} = \frac{0}{0} = \lim_{r \rightarrow 1} \frac{-10r^9}{-1} = 10$

or alternatively use the sum of a finite geometric series : $\lim_{r \rightarrow 1} \frac{1-r^{10}}{1-r} = \lim_{r \rightarrow 1} \sum_{i=0}^9 r^i = \sum_{i=0}^9 1 = 10$

4a. substitute : $u = -x^2$ $du = -2x dx$ we have $\int x e^{-x^2} dx = -\frac{1}{2} \int e^u du = -\frac{1}{2} e^u + c = -\frac{1}{2} e^{-x^2}$

4b. Integrate by parts twice to reduce the integral:

$$\begin{aligned}
&\int x^2 e^{-x} dx && u = x^2, \quad dv = e^{-x} dx \Rightarrow du = 2x dx, \quad v = -e^{-x} \\
&= -x^2 e^{-x} - 2 \int x e^{-x} dx && u = x, \quad dv = e^{-x} dx \Rightarrow du = dx, \quad v = -e^{-x} \\
&= -x^2 e^{-x} - 2x e^{-x} + 2 \int e^{-x} = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x}
\end{aligned}$$

4c. Integrate by parts: $u = x$, $dv = \sin(x) dx$, then $du = dx$ and $v = -\cos(x)$

$$\int x \sin x dx = -x \cos x + \int \cos x dx = -x \cos x + \sin x$$

4d. The easiest method is partial fractions,

$$\frac{1}{4-x^2} = \frac{1}{(2-x)(2+x)} = \frac{1/4}{2-x} + \frac{1/4}{2+x}.$$

So we get $\int \frac{dx}{4-x^2} = \frac{1}{4} \int \frac{dx}{2-x} + \frac{1}{4} \int \frac{dx}{2+x} = \frac{1}{4} [\ln(2+x) - \ln(2-x)] = \frac{1}{4} \ln \left(\frac{2+x}{2-x} \right)$.

You could also use the trig substitution $x = 2 \sin \theta$, but the algebra is much more difficult.

4e. Due to the square root we use a trig substitution, $x = 2 \sin \theta \Rightarrow dx = 2 \cos \theta d\theta \Rightarrow$

$$\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2 \cos \theta d\theta}{\sqrt{4-4 \sin^2 \theta}} = \int \frac{\cos \theta}{\sqrt{1-\sin^2 \theta}} = \int d\theta = \theta.$$

To complete the integration we need to invert the substitution to get back to the variable x . Solving our original substitution $x = 2 \sin \theta$ for θ gives $\theta = \arcsin(x/2)$, so we have $\int \frac{dx}{\sqrt{4-x^2}} = \arcsin(x/2)$.

4f. Again we use a trig substitution. Set $x = 2 \sin \theta$ so that $dx = 2 \cos \theta d\theta$. Plugging this in gives,

$$\int \sqrt{4-x^2} dx = \int \sqrt{4-4\sin^2\theta} \cdot 2\cos\theta d\theta = 4 \int \sqrt{1-\sin^2\theta} d\theta = 4 \int \cos^2\theta d\theta.$$

Now use a trig identity, $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 \Rightarrow \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$.

Inserting this into the above integral we have

$$4 \int \cos^2\theta d\theta = 2 \int (1 + \cos 2\theta) d\theta = 2\theta + \sin 2\theta = 2\theta + 2\sin\theta \cos\theta,$$

where we used the trig identity $\sin 2\theta = 2\sin\theta \cos\theta$. Now use the original substitution $x = 2\sin\theta$ to return to the original variable, $x = 2\sin\theta \Rightarrow \sin\theta = x/2 \Rightarrow \theta = \sin^{-1}(x/2)$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{1}{4}x^2}$.

This gives the final result, $\int \sqrt{4-x^2} dx = 2\sin^{-1}\frac{x}{2} + x\sqrt{1-\frac{1}{4}x^2} = 2\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2}$.

5a. For $0 \leq x \leq 1$, we have $\frac{1}{2} \leq \frac{1}{1+x} \leq 1$. Then it follows from the comparison test that

$$\int_0^1 \frac{1}{2}x^9 dx \leq \int_0^1 \frac{x^9}{1+x} dx \leq \int_0^1 x^9 dx \Rightarrow \frac{1}{2} \frac{x^{10}}{10} \Big|_0^1 \leq \int_0^1 \frac{x^9}{1+x} dx \leq \frac{x^{10}}{10} \Big|_0^1 \Rightarrow \frac{1}{20} \leq \int_0^1 \frac{x^9}{1+x} dx \leq \frac{1}{10}.$$

5b. We could expand $(1-x)^{11}$, but there is a simpler method. Set $u = 1-x$, so $x = 1-u$ and $dx = -du$, so the integral becomes

$$\begin{aligned} \int_0^1 x(1-x)^{11} dx &= \int_1^0 (1-u)u^{11}(-du) = -\int_1^0 (u^{11} - u^{12}) du = \int_0^1 (u^{11} - u^{12}) du = \frac{1}{12}u^{12} - \frac{1}{13}u^{13} \Big|_0^1 \\ &= \frac{1}{12} - \frac{1}{13} = \frac{1}{156}. \end{aligned}$$

Note that we used the property $\int_a^b f(x) dx = -\int_b^a f(x) dx$.

6. Using Hooke's Law, we have $30N = k(12 \text{ cm} - 15 \text{ cm})$, so $k = 10 \frac{\text{N}}{\text{cm}}$. To calculate the work to stretch from 12 cm (the natural length) to 20 cm, we have $W = \int_0^8 kx dx = \int_0^8 10x dx = 5x^2 \Big|_0^8 = 320 \text{ N} \cdot \text{cm} = 3.2 \text{ J}$.

7. Two ions of charge q repel each other with force $f(r) = -\frac{q^2}{4\pi\epsilon_0 r^2}$, where ϵ_0 is the vacuum permittivity, r is the distance between the ions, and the negative sign indicates that the force is repulsive. (a) If one ion is held fixed at $x = 0$, find the work W done in moving the other ion from $x = 3$ to $x = 2$. (b) Find W if these are sodium ions with charge $q = 1.5 \cdot 10^{-19} \text{ C}$ (coulomb), distance is measured in angstroms ($\text{\AA} = 10^{-10} \text{ m}$), and $(4\pi\epsilon_0)^{-1} = 9 \cdot 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$.

a) This is a problem in which an object moves from $x = a$ to $x = b$ subject to a variable force $f(x)$. In this example the coordinate x can be identified with the distance r between the two ions. Since one ion is held fixed at $x = 0$ and the second ion is moved from $x = 3$ to $x = 2$, the work done is computed as follows.

$$W = \int_3^2 -\frac{q^2}{4\pi\epsilon_0 r^2} dr = -\frac{q^2}{4\pi\epsilon_0} \int_3^2 \frac{dr}{r^2} = -\frac{q^2}{4\pi\epsilon_0} \cdot -\frac{1}{r} \Big|_3^2 = \frac{q^2}{4\pi\epsilon_0} \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{q^2}{4\pi\epsilon_0} \cdot \frac{1}{6} = \frac{q^2}{24\pi\epsilon_0}$$

$$\text{b) } W = q^2 \cdot \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{6} = (1.5 \cdot 10^{-19} \text{ C})^2 \cdot \left(9 \cdot 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2} \right) \cdot \frac{1}{6} \text{\AA}^{-1} = (1.5)^3 \cdot 10^{-29} \frac{\text{J} \cdot \text{m}}{\text{\AA}} = \frac{27}{8} \cdot 10^{-19} \text{ J}$$

8a. Draw a vertical x -axis with $x = 0$ at the base of the pyramid and $x = H$ at the top. Define $\Delta x = \frac{H}{n}$, $x_i = i\Delta x$, for $i = 0 : n$, where Δx is the width of a slice of the pyramid and x_i is the height of the i th slice. Each slice has the shape of a thin square box, so if l_i is the side length of the i th slice, then using similar triangles we see that $\frac{l_i}{H-x_i} = \frac{L}{H}$.

$$\text{volume of } i\text{th slice} = l_i^2 \cdot \Delta x = (H-x_i)^2 \frac{L^2}{H^2} \Delta x$$

$$\text{mass of } i\text{th slice} = \rho(H-x_i)^2 \frac{L^2}{H^2} \Delta x$$

$$\text{force acting on } i\text{th slice} = \rho g(H-x_i)^2 \frac{L^2}{H^2} \Delta x$$

$$\text{work done on } i\text{th slice} = \text{force} \times \text{distance} = \rho g(H-x_i)^2 \frac{L^2}{H^2} \Delta x \cdot x_i$$

$$\begin{aligned} \text{total work} = W &= \int_0^H \rho g (H-x)^2 x \frac{L^2}{H^2} dx = \rho g \frac{L^2}{H^2} \int_0^H (H-x)^2 x dx \\ \int_0^H (H-x)^2 x dx &= \int_0^H (H^2 x - 2Hx^2 + x^3) dx = \left(H^2 \frac{x^2}{2} - 2H \frac{x^3}{3} + \frac{x^4}{4} \right) \Big|_0^H = H^4 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{H^4}{12} \\ \text{so the work done in constructing the pyramid is } W &= \rho g \frac{L^2}{H^2} \cdot \frac{H^4}{12} = \frac{1}{12} \rho g L^2 H^2 \end{aligned}$$

8b. If L and H are doubled, then W increases by a factor of 16.

8c. Which requires more work, building the lower half or the upper half of the pyramid?

$$\begin{aligned} W_{\text{lower}} &: \int_0^{H/2} (H-x)^2 x dx = \left(H^2 \frac{x^2}{2} - 2H \frac{x^3}{3} + \frac{x^4}{4} \right) \Big|_0^{H/2} = H^4 \left(\frac{1}{2} \cdot \frac{1}{4} - \frac{2}{3} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{1}{16} \right) = \frac{11}{192} H^4 \\ W_{\text{upper}} &: \int_{H/2}^H (H-x)^2 x dx = \left(H^2 \frac{x^2}{2} - 2H \frac{x^3}{3} + \frac{x^4}{4} \right) \Big|_{H/2}^H = H^4 \left(\frac{1}{2} \cdot \frac{3}{4} - \frac{2}{3} \cdot \frac{7}{8} + \frac{1}{4} \cdot \frac{15}{16} \right) = \frac{5}{192} H^4 \end{aligned}$$

So more work is done building the lower half of the pyramid.

9a. $\int_1^\infty \frac{dx}{x^4}$: converges, p -test, $p = 4$; also $\int_1^\infty \frac{dx}{x^4} = \frac{1}{-3x^3} \Big|_1^\infty = 0 - \left(\frac{1}{-3} \right) = \frac{1}{3}$

9b. $\int_0^\infty x^2 e^{-x} dx$: converges

$$\begin{aligned} \int_0^\infty x^2 e^{-x} dx &= \int_0^\infty x^2 (-de^{-x}) = - \left[x^2 e^{-x} \Big|_0^\infty - \int_0^\infty e^{-x} 2x dx \right] = 2 \int_0^\infty e^{-x} x dx = -2 \int_0^\infty x de^{-x} \\ &= -2 \left[x e^{-x} \Big|_0^\infty - \int_0^\infty e^{-x} dx \right] = 2 \int_0^\infty e^{-x} dx = -2e^{-x} \Big|_0^\infty = 2 \end{aligned}$$

9c. $\int_0^\infty e^{-x} \sin x dx$: converges by comparison to e^{-x} . Also note that

$$\int_0^\infty e^{-x} \sin x dx = - \int_0^\infty e^{-x} d(\cos x) = -e^{-x} \cos x \Big|_0^\infty + \int_0^\infty -e^{-x} \cos x dx = 1 - \int_0^\infty e^{-x} \cos x dx.$$

Using the result of problem (1i), $\int_0^\infty e^{-x} \sin(x) dx = \int_0^\infty e^{-x} \cos(x) dx \Rightarrow \int_0^\infty e^{-x} \sin x dx = 1/2$.

9d. $\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1} \right) dx$: converges.

Evaluating each integral yields $\infty - \infty$, which is inconclusive, but the integral can be evaluated directly.

$$\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1} \right) dx = [\ln x - \ln(1+x)] \Big|_1^\infty = \lim_{x \rightarrow \infty} \ln \left(\frac{x}{1+x} \right) + \ln 2 = \ln 1 + \ln 2 = \ln 2$$

9e. $\int_{-r}^r \sqrt{r^2 - x^2} dx$: converges; it's a proper integral. To evaluate, use a trig substitution.

$$\begin{aligned} \sin \theta &= \frac{x}{r}, \cos \theta d\theta = \frac{dx}{r}, \sqrt{r^2 - x^2} = \sqrt{r^2 - r^2 \sin^2 \theta} = \sqrt{r^2(1 - \sin^2 \theta)} = r \cos \theta \\ \int_{-r}^r \sqrt{r^2 - x^2} dx &= \int_{-\pi/2}^{\pi/2} r \cos \theta \cdot r \cos \theta d\theta = r^2 \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta = r^2 \int_{-\pi/2}^{\pi/2} \frac{1}{2} (1 + \cos 2\theta) d\theta \\ &= \frac{r^2}{2} \cdot \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_{-\pi/2}^{\pi/2} = \frac{r^2}{2} \cdot \left(\frac{\pi}{2} + \frac{1}{2} \sin \pi - \left(-\frac{\pi}{2} + \frac{1}{2} \sin(-\pi) \right) \right) = \frac{r^2}{2} \cdot \pi = \frac{1}{2} \pi r^2 \end{aligned}$$

In fact, the graph of $f(x) = \sqrt{r^2 - x^2}$ is a semi-circle, so the integral is the area of a semi-circle.

9f. $\int_{-r}^r \frac{dx}{\sqrt{r^2 - x^2}}$: converges; it's an improper integral. To evaluate use a trig substitution.

$$\begin{aligned} \sin \theta &= \frac{x}{r}, \cos \theta d\theta = \frac{dx}{r}, \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \frac{x^2}{r^2}} = \sqrt{\frac{r^2 - x^2}{r^2}} = \frac{1}{r} \sqrt{r^2 - x^2} \\ \int_{-r}^r \frac{dx}{\sqrt{r^2 - x^2}} &= \int_{-\pi/2}^{\pi/2} \frac{r \cos \theta d\theta}{r \cos \theta} = \pi \end{aligned}$$

9g. $\int_1^\infty \frac{dx}{1+x^2}$: converges. Using the substitution $x = \tan t$ (so $dx = \sec^2 t dt$) we have

$$\int_1^\infty \frac{dx}{1+x^2} = \int_{\pi/4}^{\pi/2} \frac{\sec^2 t}{1+\tan^2 t} dt = \int_{\pi/4}^{\pi/2} 1 dt = \pi/4.$$

9h. $\int_1^\infty \frac{dx}{\sqrt{1+x^2}}$: diverges. We know this because $\frac{1}{\sqrt{1+x^2}} \sim \frac{1}{x}$ as $x \rightarrow \infty$ and the integral of $\frac{1}{x}$ diverges.

To prove it, use the comparison test.

$$\frac{1}{\sqrt{1+x^2}} \geq \frac{1}{\sqrt{x^2+x^2}} = \frac{1}{\sqrt{2}x} \text{ for } x \geq 1 \Rightarrow \int_1^\infty \frac{1}{\sqrt{1+x^2}} dx \geq \int_1^\infty \frac{1}{\sqrt{2}x} dx = \infty \quad \text{ok}$$

9i. $\int_0^\infty \frac{x}{\sqrt{x^2+1}} dx$: diverges

substituting $u = x^2 + 1, du = 2x dx$ yields $\int_0^\infty \frac{x}{\sqrt{x^2+1}} dx = \int_1^\infty \frac{du}{2\sqrt{u}} = \sqrt{u} \Big|_1^\infty = \infty - 1 = \infty$

9j. $\int_1^\infty \frac{dx}{x^2-1}$: diverges.

This can be shown by evaluating the integral using the method of partial fractions, but here we use a different approach. First note that $\frac{1}{x^2-1} \sim \frac{1}{x^2}$ as $x \rightarrow \infty$, whereas $\frac{1}{x^2-1} \sim \frac{1}{2(x-1)}$ as $x \rightarrow 1$, hence the issue is at $x = 1$, not $x = \infty$. If $1 \leq x \leq 2$, then $\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} \geq \frac{1}{3(x-1)}$.

$$\begin{aligned} \text{Then } \int_1^\infty \frac{1}{x^2-1} dx &= \int_1^2 \frac{1}{x^2-1} dx + \int_2^\infty \frac{1}{x^2-1} dx \geq \int_1^2 \frac{1}{3(x-1)} dx + \int_2^\infty \frac{1}{x^2-1} dx \\ &\geq \int_1^2 \frac{1}{3(x-1)} = \frac{1}{3} \cdot \ln(x-1) \Big|_1^2 = \frac{1}{3} \cdot (\ln 1 - \ln 0) = \infty \Rightarrow \text{the integral diverges.} \end{aligned}$$

9k. $\int_0^1 \frac{dx}{\sqrt{x}}$: converges. p -test with $p = \frac{1}{2}$; $\int_0^1 \frac{dx}{\sqrt{x}} = \int_0^1 x^{-1/2} dx = 2x^{1/2} \Big|_0^1 = 2$

9l. $\int_0^1 \frac{dx}{x^{3/2}}$: diverges. p -test, $p = \frac{3}{2}$, $\int_0^1 \frac{dx}{x^{3/2}} = \int_0^1 x^{-3/2} dx = \frac{x^{-1/2}}{-1/2} \Big|_0^1 = -2 - (-\infty) = \infty$

9m. $\int_0^1 \frac{dx}{1-x}$: diverges

substitute $u = 1 - x, du = -dx, \int_0^1 \frac{dx}{1-x} = \int_1^0 \frac{-du}{u} = \int_0^1 \frac{du}{u}$: diverges by p -test, $p = 1$

9n. substitute $u = \frac{1}{x}$, then $du = -\frac{1}{x^2} dx = -u^2 dx$ and $dx = -\frac{1}{u^2} du$

$$\begin{aligned} \int_0^\infty \frac{\ln x}{x^2+1} dx &= \int_\infty^0 \frac{\ln(1/u)}{u^{-2}+1} \frac{-du}{u^2} = -\int_\infty^0 \frac{\ln(1/u)}{1+u^2} du = \int_0^\infty \frac{\ln(1/u)}{1+u^2} du = -\int_0^\infty \frac{\ln u}{1+u^2} du \\ \Rightarrow \int_0^\infty \frac{\ln x}{x^2+1} dx &= -\int_0^\infty \frac{\ln x}{x^2+1} dx \Rightarrow \int_0^\infty \frac{\ln x}{x^2+1} dx = 0 \end{aligned}$$

10a. The total dose is $D = \int_0^\infty 2te^{-2t} dt = \int_0^\infty ye^{-y} \frac{1}{2} dy = 1/2$. The units of D are mL.

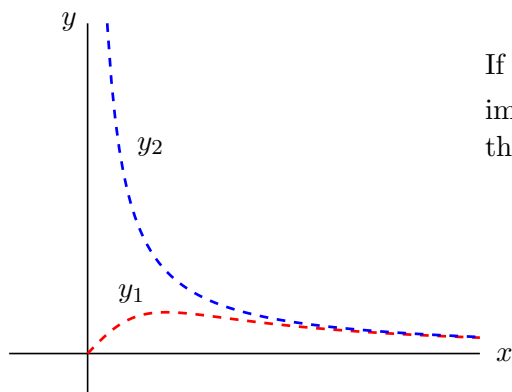
$$\begin{aligned} 10b. D_5 &= \int_0^5 2te^{-2t} dt = \int_0^{10} \frac{1}{2} ye^{-y} dy = -\frac{1}{2} \int_0^{10} y de^{-y} = -\frac{1}{2} \left[ye^{-y} \Big|_0^{10} - \int_0^{10} e^{-y} dy \right] \\ &= -\frac{1}{2} \left[-10e^{-10} + e^{-y} \Big|_0^{10} \right] = \frac{1}{2} [1 - 11e^{-10}] \Rightarrow \frac{D_5}{D} = \frac{\frac{1}{2} [1 - 11e^{-10}]}{1/2} = 0.9995 \end{aligned}$$

11. a) Show that $\frac{x}{x^2+1} \sim \frac{1}{x}$ as $x \rightarrow \infty$. b) Sketch the graphs of $y_1 = \frac{x}{x^2+1}$ and $y_2 = \frac{1}{x}$ for $x \geq 0$ on the same plot. Label each curve. Do the curves intersect?

solution : $\lim_{x \rightarrow \infty} \frac{\frac{x}{x^2+1}}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{x^2}{x^2+1} = \frac{\infty}{\infty}$, , we can proceed in two different ways as follows

method 1 : $\lim_{x \rightarrow \infty} \frac{x^2}{x^2+1} = \lim_{x \rightarrow \infty} \frac{x^2+1-1}{x^2+1} = \lim_{x \rightarrow \infty} \left(1 - \frac{1}{x^2+1}\right) = 1$ ok

method 2 : use l'Hôpital's rule , $\lim_{x \rightarrow \infty} \frac{x^2}{x^2+1} = \lim_{x \rightarrow \infty} \frac{2x}{2x} = 1$ ok



If the curves intersect, then $\frac{x}{x^2+1} = \frac{1}{x} \Leftrightarrow x^2 = x^2+1$, which is impossible, so the curves do not intersect; in fact it can be seen that $y_1 < y_2$ for all $x \geq 0$.

12. A patch of land has m_0 kg of leaves on the ground at 12 am on September 1. Each morning a fraction a of the leaves on the ground are removed, but an additional b kg of leaves fall on the ground by midnight. Let m_n be the mass of leaves on the ground at 12am on day n . a) Write the expression for m_n in terms of m_{n-1} . b) Find m_1, m_2, m_3, m_4 . c) Find the formula for m_n in terms of a, b, n, m_0 . d) Let $m_0 = 5$ kg, $a = \frac{1}{2}$ and $b = 10$ kg. Will the mass of leaves on the ground exceed 20 kg by midnight on September 30?

a) $m_n = a \cdot m_{n-1} + b$

b) $m_1 = a \cdot m_0 + b$

$m_2 = a \cdot m_1 + b = a \cdot (a \cdot m_0 + b) + b = a^2 \cdot m_0 + b(1 + a)$

$m_3 = a \cdot m_2 + b = a \cdot (a^2 \cdot m_0 + b(1 + a)) + b = a^3 \cdot m_0 + b(1 + a + a^2)$

$m_4 = a \cdot m_3 + b = a \cdot (a^3 \cdot m_0 + b(1 + a + a^2)) + b = a^4 \cdot m_0 + b(1 + a + a^2 + a^3)$

c) $m_n = a^n \cdot m_0 + b(1 + a + a^2 + a^3 + \dots + a^{n-1}) = a^n \cdot m_0 + b \cdot \frac{1 - a^n}{1 - a}$

d) No, the mass of leaves on the ground will not exceed 20 kg by midnight on September 30.

proof : $m_{30} = \left(\frac{1}{2}\right)^{30} \cdot 5 + 10 \cdot \frac{1 - \left(\frac{1}{2}\right)^{30}}{1 - \frac{1}{2}} = 5\left(\frac{1}{2}\right)^{30} + 20\left(1 - \left(\frac{1}{2}\right)^{30}\right) = 20 - 15 \cdot \left(\frac{1}{2}\right)^{30} < 20$ ok

13. a) $f(x) = \sin x, 0 \leq x \leq 2\pi$

$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{2\pi} \int_0^{2\pi} \sin x dx = \frac{1}{2\pi} \cdot -\cos x \Big|_0^{2\pi} = -\frac{1}{2\pi} (\cos 2\pi - \cos 0) = -\frac{1}{2\pi} (1 - 1) = 0$

$f_{\text{rms}}^2 = \frac{1}{b-a} \int_a^b (f(x))^2 dx = \frac{1}{2\pi} \int_0^{2\pi} \sin^2 x dx = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2} \cos 2x\right) dx$
 $= \frac{1}{2\pi} \left(\frac{1}{2}x - \frac{1}{4} \sin 2x\right) \Big|_0^{2\pi} = \frac{1}{2\pi} \cdot \frac{1}{2} 2\pi = \frac{1}{2} \Rightarrow f_{\text{rms}} = \frac{1}{\sqrt{2}}$

13. b) $f(x) = \sin 2x, 0 \leq x \leq 2\pi$

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{2\pi} \int_0^{2\pi} \sin 2x dx = \frac{1}{2\pi} \cdot \left. \frac{-\cos 2x}{2} \right|_0^{2\pi} = -\frac{1}{4\pi} (\cos 4\pi - \cos 0) = -\frac{1}{4\pi} (1 - 1) = 0$$

$$\begin{aligned} f_{\text{rms}}^2 &= \frac{1}{b-a} \int_a^b (f(x))^2 dx = \frac{1}{2\pi} \int_0^{2\pi} \sin^2 2x dx = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2} \cos 4x \right) dx \\ &= \frac{1}{2\pi} \left(\frac{1}{2}x - \frac{1}{8} \sin 4x \right) \Big|_0^{2\pi} = \frac{1}{2\pi} \cdot \frac{1}{2} 2\pi = \frac{1}{2} \Rightarrow f_{\text{rms}} = \frac{1}{\sqrt{2}} \end{aligned}$$

13. c) $f(x) = \sin^2 x, 0 \leq x \leq 2\pi$

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{2\pi} \int_0^{2\pi} \sin^2 x dx = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right) dx = \frac{1}{2\pi} \left(\frac{1}{2}x - \frac{1}{4} \sin 2x \right) \Big|_0^{2\pi} \\ &= \frac{1}{2\pi} \cdot \frac{1}{2} \cdot 2\pi = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} f_{\text{rms}}^2 &= \frac{1}{b-a} \int_a^b (f(x))^2 dx = \frac{1}{2\pi} \int_0^{2\pi} \sin^4 x dx = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right)^2 dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{4} - \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x \right) dx = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{4} - \frac{1}{2} \cos 2x + \frac{1}{4} \left(\frac{1}{2} + \frac{1}{2} \cos 4x \right) \right) dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \right) dx = \frac{1}{2\pi} \left(\frac{3}{8}x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x \right) \Big|_0^{2\pi} = \frac{1}{2\pi} \cdot \frac{3}{8} \cdot 2\pi = \frac{3}{8} \Rightarrow f_{\text{rms}} = \sqrt{\frac{3}{8}} \end{aligned}$$

14. The current and voltage in an AC circuit are $I(t) = I_0 \cos \omega t, V(t) = V_0 \cos(\omega t + \phi)$, where V_0 is peak voltage, I_0 is peak current, t is time, ω is angular frequency, and ϕ is the phase shift. The work done by the circuit in one cycle is $W = \int_0^T P(t) dt$, where $T = 2\pi/\omega$ is the period, and $P(t) = I(t) \cdot V(t)$ is the power (the rate at which the circuit does work).

a) Find the work done over 1 cycle when the current and voltage are in phase ($\phi = 0$).

$$\begin{aligned} W &= \int_0^T P(t) dt = \int_0^{2\pi/\omega} I_0 \cos \omega t \cdot V_0 \cos \omega t dt = I_0 V_0 \int_0^{2\pi/\omega} \cos^2 \omega t dt, \text{ substitute } x = \omega t \Rightarrow dx = \omega dt \\ &= I_0 V_0 \int_0^{2\pi} \cos^2 x \frac{dx}{\omega} = \frac{I_0 V_0}{\omega} \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) dx = \frac{I_0 V_0}{\omega} \left(\frac{x}{2} + \frac{1}{4} \sin 2x \right) \Big|_0^{2\pi} = \frac{I_0 V_0 \pi}{\omega} \end{aligned}$$

b) Find the work done over 1 cycle when the current and voltage are 90° out of phase ($\phi = \pi/2$).

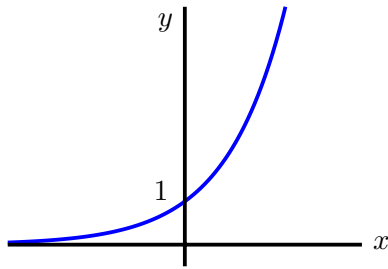
$$\phi = \frac{\pi}{2} \Rightarrow V(t) = V_0 \cos(\omega t + \phi) = V_0 \cos\left(\omega t + \frac{\pi}{2}\right) = V_0 (\cos \omega t \cos \frac{\pi}{2} - \sin \omega t \sin \frac{\pi}{2}) = -V_0 \sin \omega t$$

$$\begin{aligned} W &= \int_0^T P(t) dt = \int_0^{2\pi/\omega} I(t) \cdot V(t) dt = \int_0^{2\pi/\omega} I_0 \cos \omega t \cdot -V_0 \sin \omega t dt = -I_0 V_0 \int_0^{2\pi/\omega} \cos \omega t \sin \omega t dt \\ &= -\frac{I_0 V_0}{\omega} \int_0^{2\pi} \cos x \sin x dx = -\frac{I_0 V_0}{\omega} \int_0^{2\pi} \frac{1}{2} \sin 2x dx = -\frac{I_0 V_0}{\omega} \cdot \left. \frac{-1}{4} \cos 2x \right|_0^{2\pi} = \frac{I_0 V_0}{4\omega} (\cos 4\pi - \cos 0) = 0 \end{aligned}$$

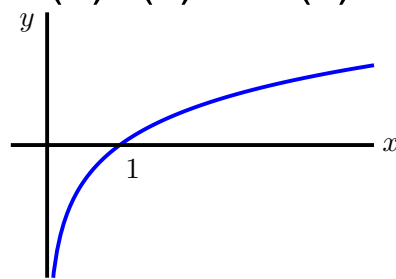
note the units : [current] = ampere, [voltage] = volt, [power] = watt = joule/second, [work] = joule

15. Sketch.

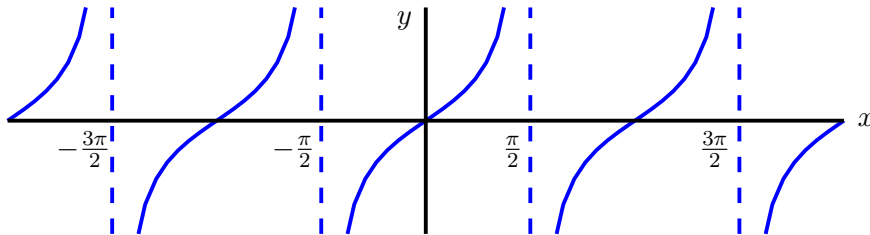
(a) $f(x) = \exp(x)$



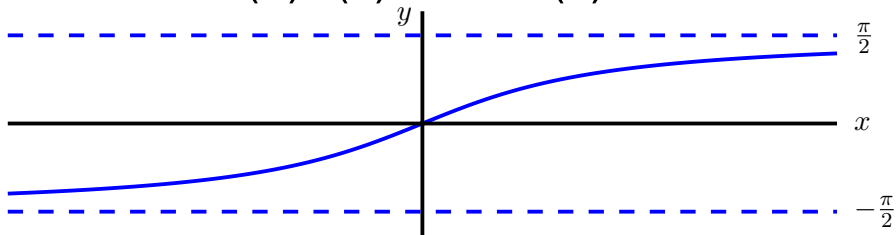
(b) $f(x) = \ln(x)$



(c) $f(x) = \tan(x)$



(d) $f(x) = \tan^{-1}(x)$



(e) $f(x) = \sin^2(x)$

