Math 156 Midterm Exam 1 Review Solutions Fall 2024

1a) True $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \Rightarrow \sum_{i=1}^{12}(2i) = 2\sum_{i=1}^{12} i = 2\frac{12\cdot 13}{2} = 156$ (can also be done by direct computation)

1b) True $\sum_{i=1}^{12}\left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{12} - \frac{1}{13}\right) = 1 - \frac{1}{13} = \frac{12}{13}$

1c) True The two sums contain the same terms in the opposite order. (This could be justified with the substitution $j = n - i$.) Alternatively, using our known formula for $\sum_{i=0}^{n} i$, we have

$$\sum_{i=0}^{n}(n-i)^2 = \sum_{i=0}^{n}(n^2 - 2in + i^2) = n^2\sum_{i=0}^{n} 1 - 2n\sum_{i=0}^{n} i + \sum_{i=0}^{n} i^2 = n^2(n+1) - 2n\frac{n(n+1)}{2} + \sum_{i=0}^{n} i^2 = \sum_{i=0}^{n} i^2.$$

1d) True In class and on homework we derived $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

Then $\left(\sum_{i=1}^{n} i\right)^2 = \left(\frac{n(n+1)}{2}\right)^2 = \sum_{i=1}^{n} i^3$.

1e) True $1 + 3 + 5 + 7 + \cdots + (2n-1) = \sum_{i=1}^{n}(2i-1) = 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} = 2\cdot\frac{n(n+1)}{2} - n = n^2$

1f) True $1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{512} = \displaystyle\sum_{i=0}^{9}\left(\frac{1}{2}\right)^i = \frac{1 - (\frac{1}{2})^{10}}{1 - \frac{1}{2}} = \frac{1 - \frac{1}{1024}}{\frac{1}{2}} = 2\left(1 - \frac{1}{1024}\right) < 2\cdot 1 = 2$

1g) True $\displaystyle\sum_{i=1}^{9}((i+1)^3 - i^3)) = (\cancel{2^3} - 1^3 + (\cancel{3^3} - \cancel{2^3}) + \cdots + (10^3 - \cancel{9^3}) = 10^3 - 1^3 = 1000 - 1 = 999$

1h) False If the number of intervals using any Riemann sum increases, then the error will decrease, not increase. In the case of the right-hand Riemann sum, if the number of intervals is doubled, the error is approximately cut in half.

1i) False $\dfrac{d}{dx}\left(\dfrac{e^{x^2}}{2x}\right) = \dfrac{2x\cdot 2xe^{x^2} - e^{x^2}\cdot 2}{4x^2} = e^{x^2}\left(\dfrac{4x^2-2}{4x^2}\right) = e^{x^2}\left(1 - \dfrac{1}{2x^2})\right) \neq e^{x^2}$

1j) True set $u = x^2, du = 2xdx$, then $\displaystyle\int xe^{x^2}dx = \int e^u\cdot\frac{1}{2}du = \frac{1}{2}\int e^u du = \frac{1}{2}e^u = \frac{1}{2}e^{x^2}$

1k) True Apply integration by parts, $\int_a^b udv = uv\big|_a^b - \int_a^b vdu$, to the first integral.

$u = e^{-x}, dv = \cos x dx \Rightarrow du = -e^{-x}dx, v = \sin x$

$\int_0^\infty e^{-x}\cos x dx = e^{-x}\sin x\big|_0^\infty + \int_0^\infty e^{-x}\sin x dx = 0 - 0 + \int_0^\infty e^{-x}\sin x dx = \int_0^\infty e^{-x}\sin x dx$

1l) True The work needed to stretch the spring from 10 cm to 15 cm is $\int_0^5 kxdx = \frac{25}{2}k = 2$ J, so $k = \frac{4}{25}$. Then the work needed to stretch the spring from 10 cm to 20 cm is $\int_0^{10}\frac{4}{25}xdx = \frac{4}{25}\cdot 50 = 8$ J.

1m) False Let x be a vertical coordinate with $x = 0$ at the top of the building and $x = L$ at the bottom of the cable. Consider a slice of the cable at position x_i. The volume of the slice is $A\Delta x$; the mass of the slice is $\rho A\Delta x$; the force acting on the slice is $\rho gA\Delta x$; the work done in pulling the slice to the top of the building is $\rho gA\Delta x\cdot x_i$. Hence the total work is $W = \lim_{n\to\infty}\rho gAx_i\Delta x = \int_0^L \rho gAxdx = \frac{1}{2}\rho gAL^2$. If the length of the cable is doubled from L to $2L$, then the work becomes $W = \frac{1}{2}\rho gA(2L)^2 = 2\rho gAL^2$, which is four times the work done in raising a cable of length L, not double.

1n) True $\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\big|_1^\infty = 0 - (-1) = 1$; the result also follows from the p-test with $p = 2$

1o) False Here's a counterexample. Let $f(x) = \frac{1}{x}, g(x) = \frac{1}{2x}$, then $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$, but $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2x}} = \lim_{x\to\infty} 2 = 2$. Of course there are many other valid counterexamples.

1p) True This follows from the comparison test.

1q) True Note that $\text{erf}(0) = 0$. Then by the FTC we have $\text{erf}'(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}$, so $\text{erf}''(x) = \frac{2}{\sqrt{\pi}}(-2x)e^{-x^2}$, and hence $\text{erf}''(0) = 0$.

1r) True

method 1: sketch the graph of each function, argue that the area is the same by symmetry method 2: use integration by parts, $u = \sin x, dv = \sin x \Rightarrow du = \cos x, v = -\cos x \Rightarrow \int_0^{\pi/2} \sin^2 x dx =$ $\sin x \cdot -\cos x\Big|_{x=0}^{\pi/2} - \int_0^{\pi/2} -\cos^2 dx = \int_0^{\pi/2} \cos^2 x dx$

method 3: use trigonometric identities, $\sin^2 x = \frac{1}{2}(1-\cos 2x), \cos^2 x = \frac{1}{2}(1+\cos 2x)$

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2. Express the integral as a limit of Riemann sums, evaluate the limit, and check by the FTC.

2a. $\int_0^2 x dx \Rightarrow a = 0, b = 2, \Delta x = \frac{b-a}{n} = \frac{2}{n}, x_i = a + i\Delta x = \frac{2i}{n}, f(x) = x, f(x_i) = x_i = \frac{2i}{n}$

Riemann sums: $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\Delta x = \lim\limits_{n\to\infty} \frac{2}{n} \sum\limits_{i=1}^{n} \frac{2i}{n} = \lim\limits_{n\to\infty} \frac{4}{n^2} \sum\limits_{i=1}^{n} i = \lim\limits_{n\to\infty} \frac{4}{n^2}\frac{n(n+1)}{2} = 2$

FTC: $\int_0^2 x dx = \frac{x^2}{2}\Big|_0^2 = 2 - 0 = 2$

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2b. $\int_0^1 x^3 dx \Rightarrow a = 0, b = 1, \Delta x = \frac{b-a}{n} = \frac{1}{n}, x_i = a + i\Delta x = \frac{i}{n}, f(x) = x^3, f(x_i) = x_i^3 = (\frac{i}{n})^3$

Riemann sums: $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\Delta x = \lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{i=1}^{n} (\frac{i}{n})^3 = \lim\limits_{n\to\infty} \frac{1}{n^4} \sum\limits_{i=1}^{n} i^3 = \lim\limits_{n\to\infty} \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2 = 1/4$

FTC: $\int_0^1 x^3 dx = \frac{x^4}{4}\Big|_0^1 = 1/4 - 0 = 1/4$

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2c. $\int_a^b x dx \Rightarrow \Delta x = \frac{b-a}{n}, x_i = a + i\Delta x, f(x) = x, f(x_i) = x_i$

$$\begin{aligned}
\text{Riemann sums}: \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} x_i \Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} (a + i\Delta x)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a + i\frac{b-a}{n}\right)\frac{b-a}{n} \\
&= \lim_{n\to\infty} \left(a\frac{b-a}{n}\sum_{i=1}^{n} 1 + \frac{(b-a)^2}{n^2}\sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty} \left(a\frac{b-a}{n}n + \frac{(b-a)^2}{n^2}\frac{n(n+1)}{2}\right) \\
&= a(b-a) + \frac{(b-a)^2}{2} = (b-a)\left(a + \frac{b-a}{2}\right) = (b-a)\left(\frac{b+a}{2}\right) = \frac{1}{2}(b^2 - a^2)
\end{aligned}$$

FTC: $\int_a^b x dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2}{2} - \frac{a^2}{2}$

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2d. $\int_a^b x^2 dx \Rightarrow \Delta x = \frac{b-a}{n}, x_i = a + i\Delta x, f(x) = x^2, f(x_i) = x_i^2$

$$\begin{aligned}
\text{Riemann sums}: \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty} \sum_{i=1}^{n} x_i^2 \Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} (a + i\Delta x)^2\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a^2 + 2ia\frac{b-a}{n} + i^2\frac{(b-a)^2}{n^2}\right)\frac{b-a}{n} \\
&= \lim_{n\to\infty} \left(a^2\frac{b-a}{n}\sum_{i=1}^{n} 1 + \frac{(b-a)^2}{n^2}2a\sum_{i=1}^{n} i + \frac{(b-a)^3}{n^3}\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty} \left(a^2\frac{b-a}{n}n + \frac{(b-a)^2}{n^2}2a\frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3}\frac{n(n+1)(2n+1)}{6}\right) \\
&= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3} = (b-a)\left(a^2 + a(b-a) + \frac{1}{3}(b-a)^2\right) \\
&= (b-a)\left(ab + \frac{1}{3}(b^2 - 2ab + a^2)\right) = \frac{1}{3}(b-a)(b^2 + ab + a^2) = \frac{1}{3}(b^3 - a^3)
\end{aligned}$$

FTC: $\int_a^b x^2 dx = \frac{x^3}{3}\Big|_a^b = \frac{b^3}{3} - \frac{a^3}{3}$

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2e. $\int_0^1 e^{-x} dx \Rightarrow a = 0, b = 1, \Delta x = \frac{b-a}{n} = \frac{1}{n}, x_i = a + i\Delta x = \frac{i}{n}, f(x) = e^{-x}, f(x_i) = e^{-x_i} = e^{-i/n}$

Riemann sums : $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} e^{-i/n} = \lim_{n\to\infty}\frac{1}{n}\left(\frac{1-\left(e^{-1/n}\right)^{n+1}}{1-e^{-1/n}} - 1\right) = \lim_{n\to\infty}\frac{1}{n}\left(\frac{1-e^{-(n+1)/n}}{1-e^{-1/n}}\right)$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1-e^{-1-1/n}}{1-e^{-1/n}}\right) = \lim_{n\to\infty}(1-e^{-1-1/n})\cdot\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1-e^{-1/n}}\right)$$

$$=(1-e^{-1})\cdot\lim_{t\to 0}\frac{t}{1-e^{-t}} = (1-e^{-1})\cdot\frac{0}{0} = (1-e^{-1})\cdot\lim_{t\to 0}\frac{1}{e^{-t}} = 1-e^{-1}$$

note: l'Hôpital's rule is used in the last step; this derivation uses the right-hand Riemann sum; the left-hand Riemann sum can also be used, but you should get the same answer.

FTC: $\int_0^1 e^{-x}dx = -e^{-x}\Big|_0^1 = -e^{-1} - (-1) = 1 - e^{-1}$

3a. $\lim_{x\to\infty} xe^{-x} = 0\cdot\infty = \lim_{x\to\infty}\frac{x}{e^x} = \frac{\infty}{\infty} = \lim_{x\to\infty}\frac{1}{e^x} = \frac{1}{\infty} = 0$; we used l'Hôpital's rule

3b. The sum is a Riemann sum for $f(x) = x^3$ on $1 \le x \le 2$. The limit is $\int_1^2 x^3 dx = [x^4/4]_1^2 = 15/4$.

3c. l'Hôpital's rule : $\lim_{x\to 0}\frac{\int_0^x f(t)\,dt}{x} = \lim_{x\to 0}\frac{f(x)}{1} = f(0)$; we assume f is continuous at 0

3d. l'Hôpital's rule : $\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\frac{e^x}{1} = 1$

3e. l'Hôpital's rule : $\lim_{r\to 1}\frac{1-r^{10}}{1-r} = \frac{0}{0} = \lim_{r\to 1}\frac{-10r^9}{-1} = 10$

or alternatively use the sum of a finite geometric series : $\lim_{r\to 1}\frac{1-r^{10}}{1-r} = \lim_{r\to 1}\sum_{i=0}^{9} r^i = \sum_{i=0}^{9} 1 = 10$

4a. substitute : $u = -x^2$ $du = -2xdx$ we have $\int xe^{-x^2}dx = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + c = -\frac{1}{2}e^{-x^2}$

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4b. Integrate by parts twice to reduce the integral:

$$\begin{array}{ll} \int x^2e^{-x}dx & u = x^2,\ dv = e^{-x}dx \Rightarrow du = 2xdx,\ v = -e^{-x} \\ = -x^2e^{-x} - 2\int xe^{-x}dx & u = x,\ dv = e^{-x}dx \Rightarrow du = dx,\ v = -e^{-x} \\ = -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x} = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} & \end{array}$$

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4c. Integrate by parts: $u = x,\ dv = \sin(x)dx$, then $du = dx$ and $v = -\cos(x)$

$\int x\sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x$

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4d. The easiest method is partial fractions,

$$\frac{1}{4-x^2} = \frac{1}{(2-x)(2+x)} = \frac{1/4}{2-x} + \frac{1/4}{2+x}.$$

So we get $\int\frac{dx}{4-x^2} = \frac{1}{4}\int\frac{dx}{2-x} + \frac{1}{4}\int\frac{dx}{2+x} = \frac{1}{4}[\ln(2+x) - \ln(2-x)] = \frac{1}{4}\ln\left(\frac{2+x}{2-x}\right)$.

You could also use the trig substitution $x = 2\sin\theta$, but the algebra is much more difficult.

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4e. Due to the square root we use a trig substitution, $x = 2\sin\theta \Rightarrow dx = 2\cos\theta d\theta \Rightarrow$

$$\int\frac{dx}{\sqrt{4-x^2}} = \int\frac{2\cos\theta d\theta}{\sqrt{4-4\sin^2\theta}} = \int\frac{\cos\theta}{\sqrt{1-\sin^2\theta}} = \int d\theta = \theta.$$

To complete the integration we need to invert the substitution to get back to the variable x. Solving our original substitution $x = 2\sin\theta$ for θ gives $\theta = \arcsin(x/2)$, so we have $\int\frac{dx}{\sqrt{4-x^2}} = \arcsin(x/2)$.

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4f. Again we use a trig substitution. Set $x = 2\sin\theta$ so that $dx = 2\cos\theta d\theta$. Plugging this in gives,

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta = 4\int \sqrt{1-\sin^2\theta}\,d\theta = 4\int \cos^2\theta\,d\theta.$$

Now use a trig identity, $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 \Rightarrow \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$.
Inserting this into the above integral we have

$$4\int \cos^2\theta\,d\theta = 2\int (1+\cos 2\theta)\,d\theta = 2\theta + \sin 2\theta = 2\theta + 2\sin\theta\cos\theta,$$

where we used the trig identity $\sin 2\theta = 2\sin\theta\cos\theta$. Now use the original substitution $x = 2\sin\theta$ to return to the original variable, $x = 2\sin\theta \Rightarrow \sin\theta = x/2 \Rightarrow \theta = \sin^{-1}(x/2)$, and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{1}{4}x^2}$.

This gives the final result, $\displaystyle\int \sqrt{4-x^2}\,dx = 2\sin^{-1}\frac{x}{2} + x\sqrt{1-\frac{1}{4}x^2} = 2\sin^{-1}\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2}$.

5a. For $0 \le x \le 1$, we have $\dfrac{1}{2} \le \dfrac{1}{1+x} \le 1$. Then it follows from the comparison test that

$$\int_0^1 \frac{1}{2}x^9 dx \le \int_0^1 \frac{x^9}{1+x}dx \le \int_0^1 x^9 dx \;\Rightarrow\; \frac{1}{2}\frac{x^{10}}{10}\Big|_0^1 \le \int_0^1 \frac{x^9}{1+x}dx \le \frac{x^{10}}{10}\Big|_0^1 \;\Rightarrow\; \frac{1}{20} \le \int_0^1 \frac{x^9}{1+x}dx \le \frac{1}{10}.$$

5b. We could expand $(1-x)^{11}$, but there is a simpler method. Set $u = 1-x$, so $x = 1-u$ and $dx = -du$, so the integral becomes

$$\int_0^1 x(1-x)^{11}dx = \int_1^0 (1-u)u^{11}(-du) = -\int_1^0 (u^{11}-u^{12})\,du = \int_0^1 (u^{11}-u^{12})\,du = \frac{1}{12}u^{12} - \frac{1}{13}u^{13}\Big|_0^1$$

$$= \frac{1}{12} - \frac{1}{13} = \frac{1}{156}.$$ Note that we used the property $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

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6. Using Hooke's Law, we have $30N = k(12\text{ cm} - 15\text{ cm})$, so $k = 10\frac{\text{N}}{\text{cm}}$. To calculate the work to stretch from 12 cm (the natural length) to 20 cm, we have $W = \int_0^8 kx dx = \int_0^8 10x dx = 5x^2\big|_0^8 = 320\text{ N}\cdot\text{cm} = 3.2\text{ J}$.

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7. Two ions each have charge q and they repel each other with force $f(r) = -\dfrac{q^2}{4\pi\epsilon_0 r^2}$, where ϵ_0 is the vacuum permittivity, r is the distance between the ions, and the negative sign indicates that the force is repulsive. (a) If one ion is held fixed at $x=0$, find the work W done in moving the other ion from $x=3$ to $x=2$. (b) Find W if these are sodium ions with charge $q = 1.5\cdot 10^{-19}$ C (coulomb), distance is measured in angstroms (Å$= 10^{-10}$ m), and $(4\pi\epsilon_0)^{-1} = 9\cdot 10^9\text{ N}\cdot\text{m}^2/\text{C}^2$.

a) This is a problem in which an object moves from $x=a$ to $x=b$ subject to a variable force $f(x)$. In this example the coordinate x can be identified with the distance r between the two ions. Since one ion is held fixed at $x=0$ and the second ion is moved from $x=3$ to $x=2$, the work done is computed as follows.

$$W = \int_3^2 -\frac{q^2}{4\pi\varepsilon_0 r^2}dr = -\frac{q^2}{4\pi\varepsilon_0}\int_3^2 \frac{dr}{r^2} = -\frac{q^2}{4\pi\varepsilon_0}\cdot -\frac{1}{r}\Big|_3^2 = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{q^2}{4\pi\varepsilon_0}\cdot\frac{1}{6} = \frac{q^2}{24\pi\varepsilon_0}$$

b) $W = q^2\cdot\dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{1}{6} = (1.5\cdot 10^{-19}\text{ C})^2\cdot\left(9\cdot 10^9\,\dfrac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\cdot\dfrac{1}{6}\,\text{Å}^{-1} = (1.5)^3\cdot 10^{-29}\,\dfrac{\text{J}\cdot\text{m}}{\text{Å}} = \dfrac{27}{8}\cdot 10^{-19}\text{ J}$

8a. Draw a vertical x-axis with $x=0$ at the base of the pyramid and $x=H$ at the top. Define $\Delta x = \frac{H}{n}, x_i = i\Delta x$, for $i = 0:n$, where Δx is the width of a slice of the pyramid and x_i is the height of the ith slice. Each slice has the shape of a thin square box, so if l_i is the side length of the ith slice, then using similar triangles we see that $\frac{l_i}{H-x_i} = \frac{L}{H}$.

volume of ith slice $= l_i^2\cdot\Delta x = (H-x_i)^2\dfrac{L^2}{H^2}\Delta x$

mass of ith slice $= \rho(H-x_i)^2\dfrac{L^2}{H^2}\Delta x$

force acting on ith slice $= \rho g(H-x_i)^2\dfrac{L^2}{H^2}\Delta x$

work done on ith slice = force × distance = $= \rho g(H-x_i)^2\dfrac{L^2}{H^2}\Delta x\cdot x_i$

total work $= W = \displaystyle\int_0^H \rho g (H-x)^2 x \frac{L^2}{H^2} dx = \rho g \frac{L^2}{H^2} \int_0^H (H-x)^2 x dx$

$$\int_0^H (H-x)^2 x dx = \int_0^H (H^2 x - 2Hx^2 + x^3) dx = \left(H^2 \frac{x^2}{2} - 2H \frac{x^3}{3} + \frac{x^4}{4} \right) \Big|_0^H = H^4 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{H^4}{12}$$

so the work done in constructing the pyramid is $W = \rho g \dfrac{L^2}{H^2} \cdot \dfrac{H^4}{12} = \dfrac{1}{12} \rho g L^2 H^2$

8b. If L and H are doubled, then W increases by a factor of 16.

8c. Which requires more work, building the lower half or the upper half of the pyramid?

$$W_{lower} : \int_0^{H/2} (H-x)^2 x dx = \left(H^2 \frac{x^2}{2} - 2H \frac{x^3}{3} + \frac{x^4}{4} \right) \Big|_0^{H/2} = H^4 \left(\frac{1}{2} \cdot \frac{1}{4} - \frac{2}{3} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{1}{16} \right) = \frac{11}{192} H^4$$

$$W_{upper} : \int_{H/2}^{H} (H-x)^2 x dx = \left(H^2 \frac{x^2}{2} - 2H \frac{x^3}{3} + \frac{x^4}{4} \right) \Big|_{H/2}^{H} = H^4 \left(\frac{1}{2} \cdot \frac{3}{4} - \frac{2}{3} \cdot \frac{7}{8} + \frac{1}{4} \cdot \frac{15}{16} \right) = \frac{5}{192} H^4$$

So more work is done building the lower half of the pyramid.

9a. $\displaystyle\int_1^\infty \frac{dx}{x^4}$: converges, p-test, $p = 4$; also $\displaystyle\int_1^\infty \frac{dx}{x^4} = \frac{1}{-3x^3} \Big|_1^\infty = 0 - \left(\frac{1}{-3} \right) = \frac{1}{3}$

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9b. $\displaystyle\int_0^\infty x^2 e^{-x} dx$: converges

$$\int_0^\infty x^2 e^{-x} dx = \int_0^\infty x^2 (-de^{-x}) = -\left[x^2 e^{-x} \Big|_0^\infty - \int_0^\infty e^{-x} 2x dx \right] = 2 \int_0^\infty e^{-x} x dx = -2 \int_0^\infty x de^{-x}$$

$$= -2 \left[x e^{-x} \Big|_0^\infty - \int_0^\infty e^{-x} dx \right] = 2 \int_0^\infty e^{-x} dx = -2e^{-x} \Big|_0^\infty = 2$$

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9c. $\displaystyle\int_0^\infty e^{-x} \sin x dx$: converges by comparison to e^{-x}. Also note that

$$\int_0^\infty e^{-x} \sin x dx = -\int_0^\infty e^{-x} d(\cos x) = -e^{-x} \cos x \Big|_0^\infty + \int_0^\infty -e^{-x} \cos x dx = 1 - \int_0^\infty e^{-x} \cos x dx.$$

Using the result of problem (1i), $\displaystyle\int_0^\infty e^{-x} \sin(x) dx = \int_0^\infty e^{-x} \cos(x) dx \Rightarrow \int_0^\infty e^{-x} \sin x dx = 1/2.$

- - -

9d. $\displaystyle\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1} \right) dx$: converges.

Evaluating each integral yields $\infty - \infty$, which is inconclusive, but the integral can be evaluated directly.

$$\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1} \right) dx = [\ln x - \ln(1+x)] \Big|_1^\infty = \lim_{x \to \infty} \ln \left(\frac{x}{1+x} \right) + \ln 2 = \ln 1 + \ln 2 = \ln 2$$

- - -

9e. $\displaystyle\int_{-r}^r \sqrt{r^2 - x^2}$: converges; it's a proper integral. To evaluate, use a trig substitution.

$\sin\theta = \frac{x}{r}, \cos\theta d\theta = \frac{dx}{r}, \sqrt{r^2 - x^2} = \sqrt{r^2 - r^2 \sin^2\theta} = \sqrt{r^2(1 - \sin^2\theta)} = r\cos\theta$

$$\int_{-r}^r \sqrt{r^2 - x^2}\, dx = \int_{-\pi/2}^{\pi/2} r\cos\theta \cdot r\cos\theta d\theta = r^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta = r^2 \int_{-\pi/2}^{\pi/2} \frac{1}{2} (1 + \cos 2\theta)\, d\theta$$

$$= \frac{r^2}{2} \cdot \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_{-\pi/2}^{\pi/2} = \frac{r^2}{2} \cdot \left(\frac{\pi}{2} + \frac{1}{2} \sin\pi - \left(-\frac{\pi}{2} + \frac{1}{2} \sin(-\pi) \right) \right) = \frac{r^2}{2} \cdot \pi = \frac{1}{2} \pi r^2$$

In fact, the graph of $f(x) = \sqrt{r^2 - x^2}$ is a semi-circle, so the integral is the area of a semi-circle.

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9f. $\displaystyle\int_{-r}^r \frac{dx}{\sqrt{r^2 - x^2}}$: converges; it's an improper integral. To evaluate se a trig substitution.

$\sin\theta = \frac{x}{r}, \cos\theta d\theta = \frac{dx}{r}, \cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{x^2}{r^2}} = \sqrt{\frac{r^2 - x^2}{r^2}} = \frac{1}{r} \sqrt{r^2 - x^2}$

$$\int_{-r}^r \frac{dx}{\sqrt{r^2 - x^2}} = \int_{-\pi/2}^{\pi/2} \frac{r\cos\theta d\theta}{r\cos\theta} = \pi$$

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9g. $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$: converges. Using the substitution $x = \tan t$ (so $dx = \sec^2 t dt$) we have

$$\int_1^\infty \frac{dx}{1+x^2} = \int_{\pi/4}^{\pi/2} \frac{\sec^2 t}{1+\tan^2 t} dt = \int_{\pi/4}^{\pi/2} 1 dt = \pi/4.$$

9h. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{1+x^2}}$: diverges. We know this because $\frac{1}{\sqrt{1+x^2}} \sim \frac{1}{x}$ as $x \to \infty$ and the integral of $\frac{1}{x}$ diverges. To prove it, use the comparison test.

$$\frac{1}{\sqrt{1+x^2}} \geq \frac{1}{\sqrt{x^2+x^2}} = \frac{1}{\sqrt{2}\,x} \text{ for } x \geq 1 \ \Rightarrow \int_1^\infty \frac{1}{\sqrt{1+x^2}} dx \geq \int_1^\infty \frac{1}{\sqrt{2}\,x} dx = \infty \quad \underline{\text{ok}}$$

9i. $\displaystyle\int_0^\infty \frac{x}{\sqrt{x^2+1}} dx$: diverges

substituting $u = x^2+1, du = 2xdx$ yields $\displaystyle\int_0^\infty \frac{x}{\sqrt{x^2+1}} dx = \int_1^\infty \frac{du}{2\sqrt{u}} = \sqrt{u}\Big|_1^\infty = \infty - 1 = \infty$

9j. $\displaystyle\int_1^\infty \frac{dx}{x^2-1}$: diverges.

This can be shown by evaluating the integral using the method of partial fractions, but here we use a different approach. First note that $\dfrac{1}{x^2-1} \sim \dfrac{1}{x^2}$ as $x \to \infty$, whereas $\dfrac{1}{x^2-1} \sim \dfrac{1}{2(x-1)}$ as $x \to 1$, hence the issue is at $x = 1$, not $x = \infty$. If $1 \leq x \leq 2$, then $\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)} \geq \dfrac{1}{3(x-1)}$.

Then $\displaystyle\int_1^\infty \frac{1}{x^2-1} dx = \int_1^2 \frac{1}{x^2-1} dx + \int_2^\infty \frac{1}{x^2-1} dx \geq \int_1^2 \frac{1}{3(x-1)} dx + \int_2^\infty \frac{1}{x^2-1} dx$

$$\geq \int_1^2 \frac{1}{3(x-1)} = \frac{1}{3} \cdot \ln(x-1)\Big|_1^2 = \frac{1}{3} \cdot (\ln 1 - \ln 0) = \infty \Rightarrow \text{the integral diverges.}$$

9k. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$: converges. p-test with $p = \frac{1}{2}$; $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \int_0^1 x^{-1/2} dx = 2x^{1/2}\Big|_0^1 = 2$

9l. $\displaystyle\int_0^1 \frac{dx}{x^{3/2}}$: diverges. p-test, $p = \frac{3}{2}$, $\displaystyle\int_0^1 \frac{dx}{x^{3/2}} = \int_0^1 x^{-3/2} dx = \frac{x^{-1/2}}{-1/2}\Big|_0^1 = -2 - (-\infty) = \infty$

9m. $\displaystyle\int_0^1 \frac{dx}{1-x}$: diverges

substitute $u = 1-x, du = -dx$, $\displaystyle\int_0^1 \frac{dx}{1-x} = \int_1^0 \frac{-du}{u} = \int_0^1 \frac{du}{u}$: diverges by p-test, $p = 1$

9n. substitute $u = \frac{1}{x}$, then $du = -\frac{1}{x^2}dx = -u^2 dx$ and $dx = -\frac{1}{u^2}du$

$$\int_0^\infty \frac{\ln x}{x^2+1} dx = \int_\infty^0 \frac{\ln(1/u)}{u^{-2}+1} \frac{-du}{u^2} = -\int_\infty^0 \frac{\ln(1/u)}{1+u^2} du = \int_0^\infty \frac{\ln(1/u)}{1+u^2} du = -\int_0^\infty \frac{\ln u}{1+u^2} du$$

$$\Rightarrow \int_0^\infty \frac{\ln x}{x^2+1} dx = -\int_0^\infty \frac{\ln x}{x^2+1} dx \Rightarrow \int_0^\infty \frac{\ln x}{x^2+1} dx = 0$$

10a. The total dose is $D = \displaystyle\int_0^\infty 2te^{-2t} dt = \int_0^\infty ye^{-y}\frac{1}{2} dy = 1/2$. The units of D are mL.

10b. $D_5 = \displaystyle\int_0^5 2te^{-2t} dt = \int_0^{10} \frac{1}{2} ye^{-y} dy = -\frac{1}{2}\int_0^{10} y de^{-y} = -\frac{1}{2}\left[ye^{-y}\Big|_0^{10} - \int_0^{10} e^{-y} dy \right]$

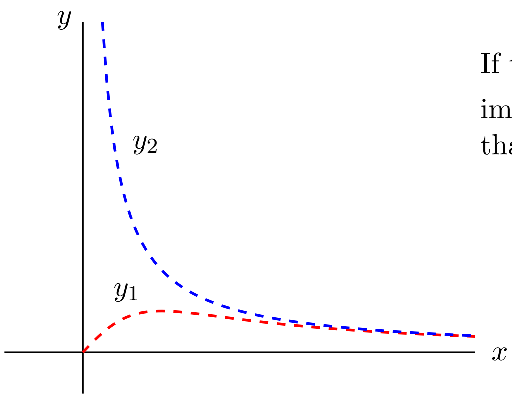
$$= -\frac{1}{2}\left[-10e^{-10} + e^{-y}\Big|_0^{10} \right] = \frac{1}{2}[1 - 11e^{-10}] \Rightarrow \frac{D_5}{D} = \frac{\frac{1}{2}[1-11e^{-10}]}{1/2} = 0.9995$$

11. a) Show that $\frac{x}{x^2+1} \sim \frac{1}{x}$ as $x \to \infty$. b) Sketch the graphs of $y_1 = \frac{x}{x^2+1}$ and $y_2 = \frac{1}{x}$ for $x \geq 0$ on the same plot. Label each curve. Do the curves intersect?

solution : $\lim_{x\to\infty} \frac{\frac{x}{x^2+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x^2}{x^2+1} = \frac{\infty}{\infty}$, , we can proceed in two different ways as follows

method 1 : $\lim_{x\to\infty} \frac{x^2}{x^2+1} = \lim_{x\to\infty} \frac{x^2+1-1}{x^2+1} = \lim_{x\to\infty} \left(1 - \frac{1}{x^2+1}\right) = 1$ ok

method 2 : use l'Hôpital's rule , $\lim_{x\to\infty} \frac{x^2}{x^2+1} = \lim_{x\to\infty} \frac{2x}{2x} = 1$ ok



If the curves intersect, then $\frac{x}{x^2+1} = \frac{1}{x} \Leftrightarrow x^2 = x^2+1$, which is impossible, so the curves do not intersect; in fact it can be seen that $y_1 < y_2$ for all $x \geq 0$.

12. A tank contains radioactive waste from a nuclear reactor. Due to radioactive decay, a fraction a of the waste in the tank at the start of a year remains in the tank at the end of the year, where $0 < a < 1$, and an additional b kg of waste is added to the tank each year. Let w_0 kg be the amount of waste in the tank on January 1, 2000, and let w_n kg be the amount of waste in the tank after n years have passed. a) Write the expression for w_n in terms of w_{n-1}. b) Find w_1, w_2, w_3, w_4. c) Find the formula for w_n in terms of a, b, n, w_0. d) Let $w_0 = 50$ kg, $a = 0.9$, and $b = 10$ kg. The capacity of the tank is 100 kg; will the waste ever exceed that value?

a) $w_n = a \cdot w_{n-1} + b$

b) $w_1 = a \cdot w_0 + b$

$w_2 = a \cdot w_1 + b = a \cdot (a \cdot w_0 + b) + b = a^2 \cdot w_0 + b(1+a)$

$w_3 = a \cdot w_2 + b = a \cdot (a^2 \cdot w_0 + b(1+a)) + b = a^3 \cdot w_0 + b(1+a+a^2)$

$w_4 = a \cdot w_3 + b = a \cdot (a^3 \cdot w_0 + b(1+a+a^2)) + b = a^4 \cdot w_0 + b(1+a+a^2+a^3)$

c) $w_n = a^n \cdot w_0 + b(1+a+a^2+a^3+\cdot+a^{n-1}) = a^n \cdot w_0 + b \cdot \frac{1-a^n}{1-a}$

d) No, the waste in the tank will never exceed 1000 kg.

proof : $w_n = (0.9)^n \cdot 50 + 10 \cdot \frac{1-(0.9)^n}{1-0.9} = (0.9)^n \cdot 50 + 10 \cdot \frac{1-(0.9)^n}{0.1} = (0.9)^n \cdot 50 + 100 \cdot (1-(0.9)^n)$

$= 100 + (0.9)^n(50-100) = 100 - 50 \cdot (0.9)^n < 100$ ok

13. a) $f(x) = \sin x, 0 \leq x \leq 2\pi$

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2\pi}\int_0^{2\pi} \sin x dx = \frac{1}{2\pi} \cdot -\cos x\Big|_0^{2\pi} = -\frac{1}{2\pi}(\cos 2\pi - \cos 0) = -\frac{1}{2\pi}(1-1) = 0$$

$$f_{\text{rms}}^2 = \frac{1}{b-a}\int_a^b (f(x))^2 dx = \frac{1}{2\pi}\int_0^{2\pi} \sin^2 x dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx$$

$$= \frac{1}{2\pi}\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)\Big|_0^{2\pi} = \frac{1}{2\pi}\cdot\frac{1}{2}2\pi = \frac{1}{2} \Rightarrow f_{\text{rms}} = \frac{1}{\sqrt{2}}$$

- -

13. b) $f(x) = \sin 2x, 0 \le x \le 2\pi$

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2\pi}\int_0^{2\pi}\sin 2x dx = \frac{1}{2\pi}\cdot\frac{-\cos 2x}{2}\Big|_0^{2\pi} = -\frac{1}{4\pi}(\cos 4\pi - \cos 0) = -\frac{1}{4\pi}(1-1) = 0$$

$$f_{\text{rms}}^2 = \frac{1}{b-a}\int_a^b (f(x))^2 dx = \frac{1}{2\pi}\int_0^{2\pi}\sin^2 2x dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 4x\right)dx$$

$$= \frac{1}{2\pi}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x\right)\Big|_0^{2\pi} = \frac{1}{2\pi}\cdot\frac{1}{2}2\pi = \frac{1}{2} \Rightarrow f_{\text{rms}} = \frac{1}{\sqrt{2}}$$

- -

13. c) $f(x) = \sin^2 x, 0 \le x \le 2\pi$

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2\pi}\int_0^{2\pi}\sin^2 x dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx = \frac{1}{2\pi}\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)\Big|_0^{2\pi}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{2}\cdot 2\pi = \frac{1}{2}$$

$$f_{\text{rms}}^2 = \frac{1}{b-a}\int_a^b (f(x))^2 dx = \frac{1}{2\pi}\int_0^{2\pi}\sin^4 x dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^2 dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\right)dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4x\right)\right)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right)dx = \frac{1}{2\pi}\left(\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x\right)\Big|_0^{2\pi} = \frac{1}{2\pi}\cdot\frac{3}{8}\cdot 2\pi = \frac{3}{8} \Rightarrow f_{\text{rms}} = \sqrt{\frac{3}{8}}$$

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14. The current and voltage in an AC circuit are $I(t) = I_0\cos\omega t, V(t) = V_0\cos(\omega t + \phi)$, where V_0 is peak voltage, I_0 is peak current, t is time, ω is angular frequency, and ϕ is the phase shift. The work done by the circuit in one cycle is $W = \int_0^T P(t)dt$, where $T = 2\pi/\omega$ is the period, and $P(t) = I(t)\cdot V(t)$ is the power (the rate at which the circuit does work).

a) Find the work done over 1 cycle when the current and voltage are in phase ($\phi = 0$).

$$W = \int_0^T P(t)dt = \int_0^{2\pi/\omega} I_0\cos\omega t\cdot V_0\cos\omega t\, dt = I_0V_0\int_0^{2\pi/\omega}\cos^2\omega t\, dt \quad , \text{ substitute } x = \omega t \Rightarrow dx = \omega dt$$

$$= I_0V_0\int_0^{2\pi}\cos^2 x\frac{dx}{\omega} = \frac{I_0V_0}{\omega}\int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)dx = \frac{I_0V_0}{\omega}\left(\frac{x}{2} + \frac{1}{4}\sin 2x\right)\Big|_0^{2\pi} = \frac{I_0V_0\pi}{\omega}$$

b) Find the work done over 1 cycle when the current and voltage are 90° out of phase ($\phi = \pi/2$).

$$\phi = \frac{\pi}{2} \Rightarrow V(t) = V_0\cos(\omega t + \phi) = V_o\cos(\omega t + \frac{\pi}{2}) = V_0(\cos\omega t\cos\frac{\pi}{2} - \sin\omega t\sin\frac{\pi}{2}) = -V_0\sin\omega t$$

$$W = \int_0^T P(t)dt = \int_0^{2\pi/\omega} I(t)\cdot V(t)dt = \int_0^{2\pi/\omega} I_0\cos\omega t\cdot -V_0\sin\omega t\, dt = -I_0V_0\int_0^{2\pi/\omega}\cos\omega t\sin\omega t\, dt$$

$$= -\frac{I_0V_0}{\omega}\int_0^{2\pi}\cos x\,\sin x\, dx = -\frac{I_0V_0}{\omega}\int_0^{2\pi}\frac{1}{2}\sin 2x\, dx = -\frac{I_0V_0}{\omega}\cdot -\frac{1}{4}\cos 2x\Big|_0^{2\pi} = \frac{I_0V_0}{4\omega}(\cos 4\pi - \cos 0) = 0$$

note the units : [current] = ampere, [voltage] = volt, [power] = watt = joule/second, [work] = joule

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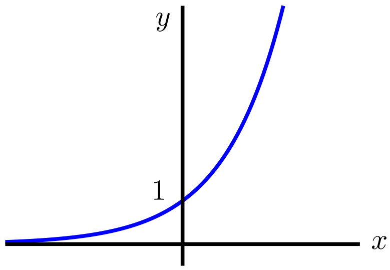
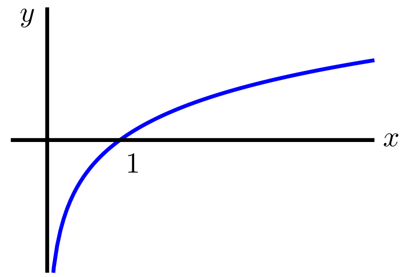
15. Sketch.

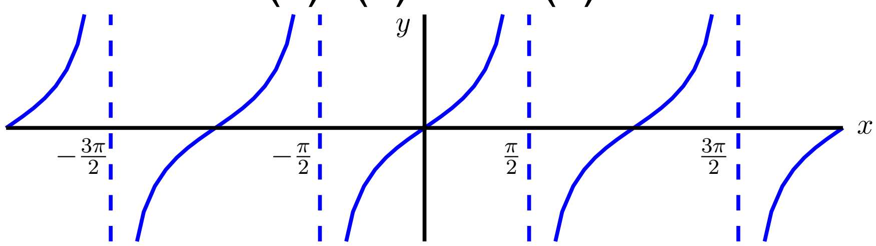
(a) $f(x) = \exp(x)$



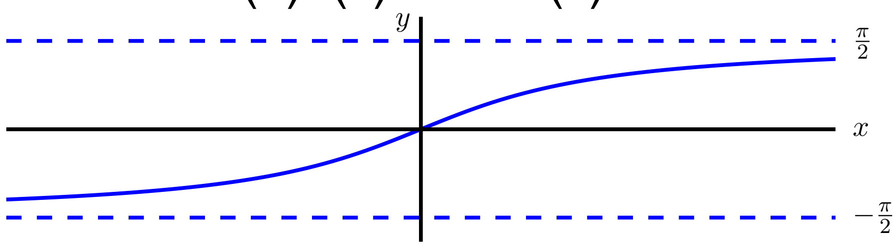
(b) $f(x) = \ln(x)$



(c) $f(x) = \tan(x)$



(d) $f(x) = \tan^{-1}(x)$



(e) $f(x) = \sin^2(x)$