For full credit, justify your answer, give units if appropriate. You may use $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

1. True or False? Justify your answer with a reason or counterexample.

a) $\int e^{x^2}dx = \frac{e^{x^2}}{2x}$ b) $\int_0^\infty x^2 e^{-x^2}dx = \frac{1}{2}\int_0^\infty e^{-x^2}dx$ c) $\lim\limits_{n\to\infty}(1-\frac{1}{n})^n = -e$

d) The center of mass of the region $\{(x,y) : 0 \le x \le 1, x^2 \le y \le \sqrt{x}\,\}$ lies on the line $y = x$.

e) If $(\overline{x}, \overline{y})$ is the center of mass of a region in the xy-plane with density $\rho = 1$, then the area of the region above the line $y = \overline{y}$ is the same as the area below the line.

f) If $f(x)$ is a probability density function, then $0 \le f(x) \le 1$ for all x.

g) The function defined by $f(x) = \begin{cases} \frac{1}{\pi\sqrt{1-x^2}} & \text{for } -1 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$ is a valid pdf.

h) If $f(x)$ is the pdf of a random variable X with mean μ, then $\text{prob}(X \le \mu) = \text{prob}(X \ge \mu)$.

i) $\int_{-\infty}^{\infty}(x-\mu)\,\frac{1}{\sigma\sqrt{2\pi}}\,e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx = 0$

j) If X is a normally distributed random variable with mean μ and standard deviation σ, then $\text{prob}(\mu - \sigma \le X \le \mu + \sigma) = \text{erf}(\frac{1}{\sqrt{2}})$, where the error function is defined by $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$.

k) The function $y(t) = \tanh t$ is a solution of the differential equation $y' = 1 - y^2$.

l) If a bacteria culture starts with 1000 cells and grows at a rate proportional to its size, and there are 2500 cells after 2 hours, then there are 4000 cells after 4 hours.

m) If a radioactive material has half-life 100 years and a sample has initial mass 1 kg, then the amount remaining after 50 years is $\frac{1}{2}$ kg.

n) If $y' = ky$ and $k > 0$, then $y(t) = 0$ is a stable constant solution of the differential equation.

o) If $y' = 1 - y^2$ and $y(0) = \frac{1}{2}$, then $\lim\limits_{t\to\infty} y(t) = 1$.

p) In solving a differential equation $y' = f(y)$ by Euler's method, if the time step Δt is reduced by a factor of $\frac{1}{2}$, then the error in the numerical solution is reduced by a factor of approximately $\frac{1}{4}$.

q) $\cosh x > \sinh x$ for all x r) $\sinh^2 x + \cosh^2 x = 1$ s) $\lim\limits_{x\to\infty}\tanh x = 0$

t) $\cosh x$ is an even function u) $\dfrac{d}{dx}\tanh x = \dfrac{1}{\cosh^2 x}$ v) $\sinh 2x = 2\sinh x\cosh x$

w) The Taylor polynomial of degree 1 for $f(x) = e^x$ at $x = 0$ is $T_1(x) = 1 + x$.

x) If $0 \le a_n \le 1$ and $a_{n+1} < a_n$, then $\lim\limits_{n\to\infty} a_n = 0$.

y) $1 = \lim\limits_{n\to\infty} 1 = \lim\limits_{n\to\infty}(n+1-n) = \lim\limits_{n\to\infty}(n+1) - \lim\limits_{n\to\infty} n = \infty - \infty = 0$

z) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is a divergent geometric series aa) $9 + 0.9 + 0.09 + 0.009 + \cdots = 10$

bb) If $\lim\limits_{n\to\infty} a_n = 0$, then $\sum\limits_{n=1}^{\infty} a_n$ converges. cc) If $\lim\limits_{n\to\infty} a_n \ne 0$, then $\sum\limits_{n=0}^{\infty} a_n$ diverges.

dd) If $0 \le a_n \le \frac{1}{n^2}$ for all n, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

ee) If $a_n = f(n)$, where $f(x)$ is positive and decreasing, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

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arclength

2. Find the arclength of the curve on the interval $0 \le x \le 1$.

a) $y = \sqrt{1-x^2}$ b) $y = \displaystyle\int_0^x \sqrt{1-t^2}\,dt$ c) $y = \dfrac{e^x + e^{-x}}{2}$ d) $y = \sqrt{x^3}$ e) $y = 2x^2$

3. Sketch the curve and find its arclength.

a) $y=\sqrt{2x-x^2}\,,\; 0\le x\le 2$ b) $y=\sqrt{x}\,,\; 0\le x\le 1$ (hint: substitute $y=\sqrt{x}$)

4. Consider the ellipse defined by $x^2+4y^2=4$ in the xy-plane. Write the perimeter of the ellipse as an integral using the formula for the arclength of a graph derived in class. Show that the substitution $x=2\sin\theta$ gives the integral in the form $8\int_0^{\pi/2}\sqrt{1-m\sin^2\theta}\,d\theta$ for some constant m. Find m. This is an elliptic integral; do not try to evaluate it; the antiderivative cannot be expressed in terms of elementary functions.

surface area

5. Sketch the curve and find the surface area obtained by rotating the curve about the x-axis.

a) $y=x^3\;,\; 0\le x\le 1$ b) $y=\sqrt{1-x}\;,\; 0\le x\le 1$ c) $y=\cosh x\;,\; 0\le x\le 1$

6. Let S be the surface area of a zone on a sphere between two parallel planes. Show that $S=2\pi rd$, where r is the radius of the sphere and d is the distance between the planes. (Hence S depends on the distance between the planes, but not on their location.)

7. An ellipse is defined by the equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where $a>b>0$. (i) Sketch the ellipse in the xy-plane. (ii) Set up the integral for the surface area S obtained by rotating the upper half of the ellipse about the x-axis. (iii) Show that $S=2\pi b\Big(\dfrac{a\sin^{-1}c}{c}+b\Big)$, where $c=\dfrac{\sqrt{a^2-b^2}}{a}$. Check your answer in the limit $a\to b$.

center of mass

8. Consider 3 point masses on the x-axis with $m_1=2, m_2=3, m_3=1$, and $x_1=-10, x_2=6$. Where should m_3 be located to ensure the CM is at $\bar{x}=0$?

9. Sketch the region, find the center of mass, and indicate the CM on the sketch.

a) $\{(x,y): 0\le x\le 1,\; 0\le y\le x\}$ b) $\{(x,y): 0\le x\le 2,\; 0\le y\le \sqrt{x(2-x)}\,\}$

c) $\{(x,y): -1\le x\le 1,\; 0\le y\le \cosh x\}$ d) $\{(x,y): -2\le x\le 2,\; y\ge 0,\; 1\le x^2+y^2\le 4\,\}$

e) $\{(x,y): 1\le x\le 2,\; 0\le y\le x^{-1}\}$

10. Consider n particles on the x-axis with mass m_i and position x_i, for $i=1,\dots,n$. Define the function $f(x)=\sum\limits_{i=1}^{n} m_i(x-x_i)^2$. Show that $f(x)$ is minimized when $x=\bar{x}$, where $\bar{x}$ is the center of mass of the particle distribution.

probability

11. The speed of cars on a highway is normally distributed with mean 55 mph and standard deviation 5 mph. Find the fraction of cars traveling between 50 mph and 60 mph.

12. The lifetime of a car battery is a random variable with pdf $f(t)=\frac{6}{625}t^2(5-t)^2$ for $0\le t\le 5$ and zero otherwise, where the time t is measured in years.

a) Sketch $f(t)$ and show that it defines a valid pdf. Find the mean battery lifetime.

b) Among 1000 batteries chosen at random, how many will last at least 3 years?

13. The time waiting in line to vote is modeled by an exponential pdf with mean 20 minutes.

a) Find the probability that a voter waits in line 10 minutes or less. (take $\sqrt{e}=1.6$)

b) Find the probability that a voter waits in line 30 minutes or more.

c) Find the median waiting time. (take $\ln 2=0.7$)

14. For a normal pdf, find the probability that the random variable lies within two standard deviations of the mean. Express the answer in terms of the error function $\mathrm{erf}(x)$.

15. Find the standard deviation of an exponential pdf with mean μ.

differential equations

16. Which of the following functions satisfy the differential equation $y''+2y'+y=0$?

a) $y=e^{-t}$ b) $y=-e^{-t}$ c) $y=2e^{-t}$ d) $y=e^{-2t}$ e) $y=te^{-t}$ f) $y=t^2e^{-t}$

17. Find the constant solutions, sketch the phase plane, and determine whether the constant solutions are stable or unstable.

a) $y' = y - 1$ b) $y' = y^2 - 1$ c) $y' = y^2 - 2y + 1$ d) $y' = y^2 - 3y + 2$ e) $y' = \sin y$

18. A particle of mass m and position $x(t)$ is moving under the influence of a force $f(x)$. Newton's 2nd law states that $x(t)$ satisfies the differential equation $mx'' = f(x)$. Let $f(x) = -V'(x)$, where $V(x)$ is the potential energy function. The total energy of the particle (kinetic + potential) is $E(t) = \frac{1}{2}mv^2 + V(x)$, where $x = x(t)$ is the particle position and $v = x'(t)$ is the particle velocity. Show that the total energy is constant in time.

19. Solve for $y(t)$ subject to initial condition $y(0) = 1$. Sketch the solution for $t \geq 0$.

a) $y' = y$ b) $y' = ty$ c) $y' = y^2$ d) $y' = y(1 - y)$

20. A tank initially contains 1000 liters of pure water. Sea water containing 0.05 kg of salt per liter enters the tank at a rate of 5 liter/min. The solution is well mixed and it drains from the tank at the same rate the sea water enters. a) Find the salt concentration in the tank after one hour. b) How much salt is present in the tank in the limit $t \to \infty$?

21. A country has \$10B in paper currency in circulation at any time. There are two types of bills, old-type and new-type. Each day the treasury releases \$50M of new-type bills into circulation and \$50M goes out of circulation into the banks due to business transactions. Assume that when the new bills are released each day, they are instantaneously mixed with the currency in circulation. Let $x(t)$ denote the value of all new-type bills in circulation at time t and assume $x(0) = 0$. Use \$1B as the unit of currency and 1 day as the unit of time. a) Write down the differential equation for $x(t)$. b) Solve for $x(t)$. c) How long will it take for the new-type bills to reach 90% of the total currency in circulation? (take $\ln 10 = 2.3$)

22. In a chemical reactor, one molecule of type A and one molecule of type B combine to form one molecule of type C, A + B $\to$ C. Let a_0, b_0 be the initial concentrations of the reactants A, B, and let $c(t)$ be the concentration of the product C at time t. The <u>law of mass action</u> says that the reaction rate is proportional to the product of the reactant concentrations, $c' = k(a_0 - c)(b_0 - c)$, where $k > 0$. Take $a_0 = 1$ mole/L, $b_0 = 2$ mole/L, and sketch the phase plane of the system. Find the product concentration $c(t)$ assuming there is no product initially present in the reactor. Find the value of $c(t)$ in the limit $t \to \infty$. Would this value change if the initial product concentration was 1.5 mole/L instead of zero?

23. When an object is heated to a high temperature T_0 and then removed from the heat source, the object's temperature $T(t)$ decays by radiation according to the equation, $mc_PT' = -e\sigma T^4$, where m is the object's mass, c_P is the specific heat capacity, e is the emissivity, and σ is Stefan's constant. Assume that m, c_P, e, σ are all positive. Find the temperature $T(t)$ and sketch the graph for $t \geq 0$. In the limit $t \to \infty$, the temperature $T(t)$ decays like $t^{-\alpha}$; find α.

24. A radioactive sample has mass 128 kg after two hours and mass 2 kg after five hours. (a) What was the initial mass of the sample? (b) When will the sample mass decay to 1 kg? Express the answer in simplest form.

25. A \$500 investment has a 12% annual interest rate. Find the investment value after 10 years if it is a) compounded annually, b) compounded continuously. c) Find the equivalent annual interest rate if the interest is compounded continuously.

26. A thermometer is taken outside from an air-conditioned room where the temperature is 21°C. It reads 27°C after one minute and 30°C after two minutes. Find the outdoor temperature.

27. The university endowment receives donations at a constant rate r [dollar/year] and a certain amount is spent at a rate proportional to the endowment size with proportionality constant k [1/year]. Assume $r > 0, k > 0$. Let $y(t)$ [dollar] denote the endowment value at time t [year].

a) Which differential equation expresses the rate of change of the endowment in time?

i) $y' = ry + k$ ii) $y' = r + ky$ iii) $y' = r - ky$ iv) $y' = ky - r$

b) Let C be the constant solution of the equation. Find C in terms of r and k. Sketch the phase plane of the differential equation. Is the constant solution stable or unstable?

c) The endowment drops to $\frac{1}{2}C$ in a recession. How long will it take to recover to $\frac{3}{4}C$? Assume that the spending proportionality constant is $k = 0.05$ and take $\ln 2 \approx 0.7$.

28. Bob and Ray each order 8 oz cups of coffee which are served hot at temperature T_h and they use different strategies to cool the coffee. Bob immediately adds 1 oz of cold milk at temperature $T_c < T_h$ and then waits 2 minutes before drinking, while Ray waits 2 minutes and then adds 1 oz of cold milk. Assume the ambient temperature T_a in the cafe satisfies $T_c < T_a < T_h$. Who ends up drinking cooler coffee, Bob or Ray? You may explain your answer intuitively, but for full credit you should justify it by finding formulas for $T_{Bob}(t), T_{Ray}(t)$ in terms of T_c, T_a, T_h and k, where k is the rate constant for cooling a cup of coffee.

29. Assume the rate at which a rumor spreads is proportional to the product of two terms, the fraction of the population who have already heard the rumor and the fraction of the population who have not yet heard the rumor. Consider a town with 1000 inhabitants. Suppose 10 people have heard a rumor at 8am and 20 people have heard it at 9am. At what time will half the population have heard the rumor?

30. A 10 kg package with a parachute is dropped from an airplane at altitude 1 km. The package falls under the influence of two forces, the force of gravity (take $g = 10\ \mathrm{m/s^2}$), and a drag force proportional to the square of the package velocity with proportionality constant $k = 4$ kg/m. How long does it take the package to reach the ground? (hint 1: Newton's 2nd law gives a 2nd order differential equation for position $y(t)$; rewrite it as a 1st order differential equation for velocity $v(t) = y'(t)$; hint 2: show that $\int \frac{dx}{1-x^2} = \tanh^{-1}(x)$ by the substitution $x = \tanh u$.)

series

31. Find the sum of the series. (hint for (c): group each set of three consecutive terms)

a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ b) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$ c) $1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$

32. Consider the series $\sum_{n=0}^{\infty} (2x-1)^n$, where x is a variable. (a) For what values of x does the series converge? (b) Let $f(x)$ be the sum of the series; find a formula for $f(x)$; sketch the graph for $-5 \le x \le 5$; indicate on the graph the interval on the x-axis where the series converges.

33. Determine whether the series converges or diverges. a) $\sum_{n=1}^{\infty} \frac{n}{n+1}$ b) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ c) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

hyperbolic functions

34. Find the antiderivative. a) $\int \cosh 2x\, dx$ b) $\int \tanh 2x\, dx$ c) $\int \cosh^2 x\, dx$

35. Verify the following addition formulas for sinh and cosh. (hint: first derive (a) directly using the definitions, then (b), (c), (d) can be derived from (a) with relatively little extra work)

(a) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$ (b) $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

(c) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$ (d) $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

36. a) Find $\lim_{x\to\pm\infty} \operatorname{sech} x$. b) Find $\frac{d}{dx}\operatorname{sech} x$. c) Sketch $y = \operatorname{sech} x$ for $-\infty < x < \infty$.

37. Show that $\sinh x \sim \cosh x$ as $x \to \infty$. 38. Show that $\frac{1}{2}\ln\frac{1+x}{1-x} = \tanh^{-1} x$.

Taylor approximation

39. For each function $f(x)$ and point a below, find $T_1(x)$, the linear Taylor approximation for $f(x)$ at the given point. In each case, sketch the graphs of $f(x)$ and $T_1(x)$ on the same plot, and indicate the point $(a, f(a))$ on the graph.

a) $f(x) = x^{-1},\ a = 1$ b) $f(x) = \sin x,\ a = \frac{\pi}{2}$ c) $f(x) = \sqrt{x},\ a = 4$

In (c), setting $x = 5$ yields $f(5) = \sqrt{5} = 2.2361$; compute the approximation $T_1(5)$.

miscellaneous

40. On homework you showed that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$. Now find $\lim_{n\to\infty} n\left(\left(1 + \frac{1}{n}\right)^n - e\right)$.

This confirms a result about Euler's method on page 48 of the lecture notes.