Math 156 Applied Honors Calculus II Review Sheet Solutions for Midterm Exam 2 Fall 2023

1. True or False? Justify your answer with a reason or counterexample.

a) False. use the quotient rule to compute the derivative

$$\frac{d}{dx}\frac{e^{x^2}}{2x} = \frac{2x\cdot 2xe^{x^2} - e^{x^2}\cdot 2}{(2x)^2} = \frac{(4x^2-2)e^{x^2}}{4x^2} = \frac{(2x^2-1)e^{x^2}}{2x^2} \neq e^{x^2} \Rightarrow \int e^{x^2}dx \neq \frac{e^{x^2}}{2x}$$

- -

b) True. use integration by parts : $u = x, dv = xe^{-x^2} \Rightarrow du = dx, v = -\frac{1}{2}e^{-x^2}$

$$\int_0^\infty x^2e^{-x^2}dx = x(-\tfrac{1}{2}e^{-x^2})\Big|_0^\infty - \int_0^\infty -\tfrac{1}{2}e^{-x^2}\,dx = \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

- -

c) False. $\lim_{n\to\infty}(1-\frac{1}{n})^n = L \Rightarrow \ln L = \ln \lim_{n\to\infty}(1-\frac{1}{n})^n = \lim_{n\to\infty}\ln(1-\frac{1}{n})^n = \lim_{n\to\infty} n\ln(1-\frac{1}{n}) = \infty\cdot 0$

set $u = \frac{1}{n} \Rightarrow \ln L = \lim_{u\to 0}\frac{\ln(1-u)}{u} = \frac{0}{0} = \lim_{u\to 0}\frac{\frac{-1}{1-u}}{1} = -1 \Rightarrow L = e^{-1}$

- -

d) True. The region is symmetric about the line $y = x$ and hence by the symmetry principle the CM lies on the line $y = x$.

- -

e) False. A counterexample was given in class on page 34 of the lecture notes. A triangular plate has vertices in the xy-plane at $(0,0),(1,0),(0,1)$. Then $\bar{x} = \frac{1}{3}$. However the area to the left of the line $x = \bar{x}$ is $\frac{5}{18}$ and the area to the right is $\frac{4}{18}$.

- -

f) False. A counterexample is the normal distribution $f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Then $f(\mu) = \frac{1}{\sqrt{2\pi}\sigma} > 1$ for $\sigma < \frac{1}{\sqrt{2\pi}}$.

- -

g) True. 1. $f(x) \geq 0$ for all x

2. $\int_\infty^\infty f(x)dx = \int_{-1}^1 \frac{1}{\pi\sqrt{1-x^2}}dx$, set $x = \sin\theta, dx = \cos\theta d\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi\sqrt{1-\sin^2\theta}}\cos\theta d\theta = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos\theta}{\cos\theta}d\theta = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \frac{\theta}{\pi}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1$$

- -

h) False. The median m satisfies prob$(X \leq m) =$ prob$(X \geq m)$. In general, $\mu \neq m$ and prob$(X \leq \mu) \neq$ prob$(X \geq \mu)$, unless the pdf is symmetric about $x = \mu$.

- -

i) True. The pdf of a normal distribution is $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}$ and we know from hw7 that $\int_{-\infty}^\infty f(x)dx = 1, \int_{-\infty}^\infty xf(x)dx = \mu$. So the given integral is $\int_{-\infty}^\infty \frac{1}{\sigma\sqrt{2\pi}}(x-\mu)e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \int_{-\infty}^\infty (x-\mu)f(x)dx = \int_{-\infty}^\infty xf(x)dx - \mu\int_{-\infty}^\infty f(x)dx = \mu - \mu = 0$.

- -

j) True. If X is a normally distributed random variable with mean μ and standard deviation σ, then prob$(\mu - \sigma \leq X \leq \mu + \sigma) = \int_{\mu-\sigma}^{\mu+\sigma}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$, set $u = \frac{x-\mu}{\sqrt{2}\sigma}$, $du = \frac{dx}{\sqrt{2}\sigma}$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\frac{1}{\sigma\sqrt{2\pi}}e^{-u^2}\sqrt{2}\sigma dy = \frac{1}{\sqrt{\pi}}\int_{-1/\sqrt{2}}^{1/\sqrt{2}}e^{-u^2}du = \frac{2}{\sqrt{\pi}}\int_0^{1/\sqrt{2}}e^{-u^2}du = \text{erf}(\tfrac{1}{\sqrt{2}}),$$

where the error function is defined by $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$.

- -

k) True. $y(t) = \tanh t \Rightarrow y'(t) = \frac{d}{dt}\tanh t = \frac{d}{dt}\frac{\sinh t}{\cosh t} = \frac{\cosh^2 t - \sinh^2 t}{\cosh^2 t} = 1 - \tanh^2 t = 1 - y^2$

- -

l) False. See page 42 of the lecture notes. When a population grows at a rate proportional to itself, the population increases at an exponential rate, not a linear rate. The cell count is $y(t) = 1000(2.5)^{t/2}$, so $y(4) = 6250$ cells.

- -

m) False. If a radioactive material has half-life 100 years and a sample has initial mass 1 kg, then the amount remaining after 100 years is $\frac{1}{2}$ kg and the amount remaining after only 50 years is more than $\frac{1}{2}$ kg.

- -

n) False. $y(t) = 0$ is a constant solution, but it is unstable because if $y_0 \neq 0$, then $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y_0 e^t \neq 0$.

- -

o) True. If $y' = 1 - y^2$, then the constant solutions are $y = \pm 1$, and the phase plane shows that $y = 1$ is stable and $y = -1$ is unstable, so if $y(0) = \frac{1}{2}$, then $\lim_{t\to\infty} = 1$ since $y = 1$ is stable.

- -

p) False. As shown in class and on homework, if Δt is reduced by a factor of $\frac{1}{2}$, then the error in Euler's method decreases by a factor of approximately $\frac{1}{2}$, not $\frac{1}{4}$.

- -

q) True. $\cosh x - \sinh x = \frac{e^x+e^{-x}}{2} - \frac{e^x-e^{-x}}{2} = e^{-x} > 0$ for all x

- -

r) False. $\sinh^2 x + \cosh^2 x = (\frac{e^x-e^{-x}}{2})^2 + (\frac{e^x+e^{-x}}{2})^2 = \frac{e^{2x}-2+e^{-2x}}{4} + \frac{e^{2x}+2+e^{-2x}}{4} = \frac{e^{2x}+e^{-2x}}{2} = \cosh 2x \neq 1$ for $x \neq 0$

- -

s) False. $\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{\sinh x}{\cosh x} = \lim_{x\to\infty} \frac{\frac{e^x-e^{-x}}{2}}{\frac{e^x+e^{-x}}{2}} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{\frac{1-e^{-2x}}{2}}{\frac{1+e^{-2x}}{2}} = 1$

- -

t) True. By definition $\cosh x = \frac{e^x+e^{-x}}{2}$, so $\cosh(-x) = \frac{e^{-x}+e^x}{2} = \cosh x$, so $\cosh x$ is an even function.

- -

u) True. $\frac{d}{dx} \tanh x = \frac{d}{dx} \frac{\sinh x}{\cosh x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$

- -

v) True. $2 \sinh x \cosh x = 2(\frac{e^x-e^{-x}}{2})(\frac{e^x+e^{-x}}{2}) = \frac{1}{2}(e^{2x} - e^{-2x}) = \sinh(2x)$

- -

w) True. $T_1(x) = f(a) + f'(a)(x-a)$, then $f(x) = e^x, a = 0 \Rightarrow f(0) = 1, f'(0) = 1 \Rightarrow T_1(x) = 1 + x$

- -

x) False. counterexample : $a_n = \frac{1}{2} + \frac{1}{2n} \Rightarrow 0 \leq a_n \leq 1$ and $a_{n+1} < a_n$, but $\lim_{n\to\infty} a_n = \frac{1}{2} \neq 0$
The two given conditions imply the sequence converges, but the limit is not necessary zero.

- -

y) False. The 3rd equal sign is not justified, i.e. $\lim_{n\to\infty}(n+1-n) \neq \lim_{n\to\infty}(n+1) - \lim_{n\to\infty} n$, because the left side is equal to 1, but the right side is undefined.

- -

z) False. This is the harmonic series; it diverges, but it is not a geometric series.

- -

aa) True. $9 + 0.9 + 0.09 + 0.009 + \cdots = 9(1 + 0.1 + 0.001 + 0.001 + \cdots) = 9 \sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n = 9 \cdot \frac{1}{1-\frac{1}{10}} = 10$

- -

bb) False. counterexample : $a_n = \frac{1}{n}$, then $\lim_{n\Rightarrow\infty} a_n = \lim_{n\Rightarrow\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

- -

cc) True. We showed that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$; so if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

- -

dd) True. The comparison test for series implies that $0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$, and the latter series converges by the p-test with $p = 2$, hence the sum of the a_n series converges.

- -

ee) False. The function $f(x)$ satisfies the assumptions in the integral test for series, but the series converges if and only if $\int_1^\infty f(x)dx$ converges and that is not true in general.

2a. $y = \sqrt{1-x^2}$, $0 \le x \le 1$. So $L = \displaystyle\int_0^1 \sqrt{1 + \left(\frac{x}{\sqrt{1-x^2}}\right)^2}dx = \int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \pi/2.$

The final integral is evaluated by trig substitution, $x = \sin\theta, dx = \cos\theta d\theta$.

- -

2b. $y = \displaystyle\int_0^x \sqrt{1-t^2}\,dt \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\int_0^x \sqrt{1-t^2}\,dt = \sqrt{1-x^2}$. Plugging this in yields,

$$L = \int_0^1 \sqrt{1+(1-x^2)}\,dx = \int_0^1 \sqrt{2-x^2}\,dx = \frac{x}{\sqrt{2}}\sqrt{1-(x/\sqrt{2})^2} + \arcsin(x/\sqrt{2})\Big|_0^1 = 1/2 + \pi/4.$$

- -

2c. $$L = \int_0^1 \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}dx = \int_0^1 \sqrt{1 + \frac{e^{2x} - 2 + e^{-2x}}{4}}dx = \int_0^1 \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}}dx$$
$$= \int_0^1 \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}dx = \int_0^1 \frac{e^x + e^{-x}}{2}dx = \frac{e^x - e^{-x}}{2}\Big|_0^1 = \frac{e - e^{-1}}{2}$$

- -

2d. $f'(x) = \dfrac{3}{2}x^{\frac{1}{2}}$

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx = \int_0^1 \sqrt{1+\frac{9}{4}x}\,dx = \frac{4}{9}\int_1^{13/4} u^{1/2}du = \frac{4}{9}\frac{u^{3/2}}{3/2}\Big|_1^{13/4} = \frac{8}{27}\left(\left(\frac{13}{4}\right)^{3/2} - 1\right)$$

- -

2e. $f'(x) = 4x \Rightarrow L = \displaystyle\int_a^b \sqrt{1+(f'(x))^2}dx = \int_0^1 \sqrt{1+16x^2}dx.$

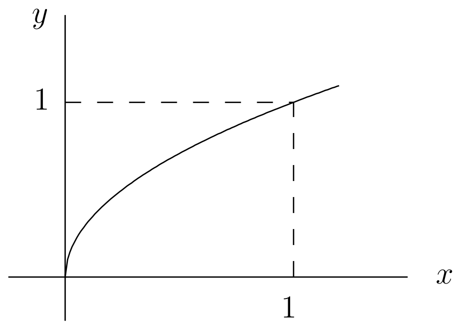
Substitute $\tan\theta = 4x; \sec^2\theta\; d\theta = 4\; dx$, leading to

$$\begin{aligned} L &= \int_{x=0}^{x=1} \sqrt{1+\tan^2\theta}\frac{1}{4}\sec^2\theta\; d\theta = \frac{1}{4}\int_{x=0}^{x=1} \sec^3\theta\; d\theta \\ &= \frac{1}{8}(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta))]_{x=0}^{x=1} = \frac{1}{8}(4x\sqrt{1+16x^2} + \ln(\sqrt{1+16x^2} + 4x))]_0^1 \\ &= \frac{1}{8}(4\sqrt{17} + \ln(4+\sqrt{17})) \end{aligned}$$

3a. The curve $y = \sqrt{2x - x^2}$ is, by completing the square, $y = \sqrt{1-(x-1)^2}$. This is the equation of the upper semicircle of radius 1 centered at $(1, 0)$. Therefore, without doing any calculus, the arclength is $\frac{1}{2}2\pi r = \pi$. Note that the arclength can also be computed using the arclength integral.

- -

3b. The curve $y = \sqrt{x}$ for $0 \le x \le 1$ is the inverse of a parabola.



$f(x) = \sqrt{x} \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x}} \Rightarrow L = \displaystyle\int_a^b \sqrt{1 + (f'(x))^2}dx = \int_0^1 \sqrt{1 + \frac{1}{4x}}dx$: improper, converges

substitute : $y = \sqrt{x} \Rightarrow dy = \dfrac{dx}{2\sqrt{x}} = \dfrac{dx}{2y}$

$L = \displaystyle\int_0^1 \sqrt{1 + \frac{1}{4y^2}} \cdot 2y\, dy = \int_0^1 \sqrt{4y^2 + 1}\, dy =$ arclength of a parabola $= \frac{1}{4}(2\sqrt{5} + \ln(2 + \sqrt{5}))$

---

4. $x^2 + 4y^2 = 4 \Rightarrow 4y^2 = 4 - x^2 \Rightarrow y^2 = 1 - x^2/4 \Rightarrow y = \sqrt{1 - x^2/4}$

$f(x) = (1 - x^2/4)^{1/2} \Rightarrow f'(x) = \dfrac{1}{2}(1 - x^2/4)^{-1/2} \cdot -\dfrac{x}{2} = \dfrac{-x}{4\sqrt{1 - x^2/4}} = \dfrac{-x}{\sqrt{16 - 4x^2}}$

$1 + (f'(x))^2 = 1 + \dfrac{x^2}{16 - 4x^2} = \dfrac{16 - 4x^2 + x^2}{16 - 4x^2} = \dfrac{16 - 3x^2}{16 - 4x^2}$

$S = \displaystyle\int_a^b \sqrt{1 + (f'(x))^2}dx = 2\int_{-2}^2 \sqrt{\frac{16 - 3x^2}{16 - 4x^2}}dx = 4\int_0^2 \sqrt{\frac{16 - 3x^2}{16 - 4x^2}}dx$, set $x = 2\sin\theta, dx = 2\cos\theta d\theta$

$= 4\displaystyle\int_0^{\pi/2} \frac{\sqrt{16 - 3 \cdot 4\sin^2\theta}}{4\cos\theta} \cdot 2\cos\theta d\theta = 2\int_0^{\pi/2} \sqrt{16 - 12\sin^2\theta}d\theta = 8\int_0^{\pi/2} \sqrt{1 - \frac{3}{4}\sin^2\theta}d\theta \Rightarrow m = \frac{3}{4}$

5a. $S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}dx = \int_0^1 2\pi x^3\sqrt{1 + (3x^2)^2}dx = \int_0^1 2\pi\sqrt{1 + 9x^4}x^3dx$
set $u = 1 + 9x^4$, so $du = 9 \cdot 4x^3dx$, and $x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 10$, then we have
$S = \int_1^{10} 2\pi\sqrt{u}\frac{1}{9\cdot 4}du = \frac{1}{18}\pi\int_1^{10}\sqrt{u}du = \frac{1}{18}\pi\frac{2}{3}u^{\frac{3}{2}}\Big|_1^{10} = \frac{\pi}{27}(\sqrt{10}^3 - 1)$

- -

5b. $S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}dx = \int_0^1 2\pi\sqrt{1 - x}\sqrt{1 + (\frac{-1}{2\sqrt{1-x}})^2}dx$
$= \int_0^1 2\pi\sqrt{1 - x}\sqrt{1 + \frac{1}{4(1-x)}}dx = \int_0^1 2\pi\sqrt{1 - x}\sqrt{\frac{5-4x}{4(1-x)}}dx$
$= \int_0^1 \pi\sqrt{5 - 4x}dx = \int_0^1 \pi\sqrt{5 - 4x}(-\frac{1}{4})d(5 - 4x) = -\frac{\pi}{4}\frac{2}{3}(5 - 4x)^{\frac{3}{2}}\Big|_0^1$
$= -\frac{\pi}{6} + \frac{\pi}{6}(\sqrt{5})^3 = \frac{\pi}{6}((\sqrt{5})^3 - 1)$

- -

5c. $S = 2\pi\int_0^1 f(x)\sqrt{1 + (f'(x))^2}\, dx = 2\pi\int_0^1 \cosh x\sqrt{1 + \sinh^2 x}\, dx = 2\pi\int_0^1 \cosh^2 x\, dx$
$= 2\pi\int_0^1 \left(\frac{e^x+e^{-x}}{2}\right)^2 dx = 2\pi\int_0^1 \left(\frac{e^{2x}+2+e^{-2x}}{4}\right) dx = \frac{\pi}{2}\left(\frac{e^{2x}}{2} + 2x + \frac{e^{-2x}}{-2}\right)\Big|_0^1$
$= \frac{\pi}{2}\left(\frac{e^2}{2} + 2 - \frac{e^{-2}}{2} - \left(\frac{1}{2} - \frac{1}{2}\right)\right) = \frac{\pi}{2}(2 + \sinh 2) = \pi\left(1 + \frac{1}{2}\sinh 2\right)$

---

6. Assume the two planes are in the right half of the xy-plane, $x > 0$; otherwise we can either perform a similar calculation for the left half plane, or if they are in both, perform two calculations, one for the left half plane and another for the right half plane. A sphere is generated by rotating the curve $f(x) = \sqrt{r^2 - x^2}$ around the x axis. The formula for the surface area of the zone is
$S = 2\pi\int_a^b f(x)\sqrt{1 + (f'(x))^2}dx$, where a, b are the appropriate limits.

Here we have $1+(f'(x))^2 = \frac{r^2}{r^2-x^2}$, so that $S = 2\pi\int_a^b \sqrt{r^2-x^2}\frac{r}{\sqrt{r^2-x^2}}\mathrm{d}x = 2\pi r(b-a)$.
Letting $d = b-a$ be the distance between the planes, we find that $S = 2\pi rd$, and hence the surface area of the zone depends only on the distance between the planes and not their location.

---

7. (ii) $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \Rightarrow y = \pm\sqrt{b^2-\left(\frac{b}{a}\right)^2 x^2}$ (use the upper half of the ellipse)

$f(x) = \frac{b}{a}\sqrt{a^2-x^2} \Rightarrow f'(x) = -\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}} \Rightarrow \sqrt{1+(f'(x))^2} = \sqrt{1+\left(\frac{b}{a}\right)^2\frac{x^2}{a^2-x^2}} = \frac{\sqrt{a^2-x^2+\left(\frac{b}{a}\right)^2x^2}}{\sqrt{a^2-x^2}}$

$S = 2\pi\int_{-a}^a f(x)\sqrt{1+(f'(x))^2}dx = 2\pi\int_{-a}^a \frac{b}{a}\sqrt{a^2-x^2}\frac{\sqrt{a^2-x^2+\left(\frac{b}{a}\right)^2x^2}}{\sqrt{a^2-x^2}}dx$

$= 2\pi\frac{b}{a}\int_{-a}^a \sqrt{a^2+\left(\frac{b^2-a^2}{a^2}\right)x^2}dx = 4\pi\frac{b}{a}\int_0^a \sqrt{a^2+\left(\frac{b^2-a^2}{a^2}\right)x^2}dx$ (the integrand is even)

- -

(iii) Now we will show that $S = 2\pi b(b+a(\sin^{-1}c)/c)$, where $c = \sqrt{a^2-b^2}/a$.

$S = 4\pi\frac{b}{a}\int_0^a \sqrt{a^2+\left(\frac{b^2-a^2}{a^2}\right)x^2}dx = 4\pi\frac{b}{a}\int_0^a \sqrt{a^2-\left(\frac{a^2-b^2}{a^2}\right)x^2}dx$

$= 4\pi\frac{b}{a}\int_0^a \sqrt{a^2-\left(\frac{\sqrt{a^2-b^2}}{a}x\right)^2}dx = 4\pi\frac{b}{a}\int_0^a \sqrt{a^2-(cx)^2}dx$

substitute $u = cx$, $du = cdx$, change limits $x = a \to u = ca, x = 0 \to u = 0$

$S = 4\pi\frac{b}{a}\int_0^{ca}\sqrt{a^2-u^2}\,\frac{du}{c} = 4\pi\frac{b}{ac}\left[\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)\right]_0^{ca}$ (this step uses trig substitution)

$= 2\pi\frac{b}{ac}\left(ca\sqrt{a^2-c^2a^2}+a^2\sin^{-1}c\right) = 2\pi b\left(\sqrt{a^2-c^2a^2}+\frac{a}{c}\sin^{-1}c\right)$

note : $\sqrt{a^2-c^2a^2} = \sqrt{a^2-\left(\frac{a^2-b^2}{a^2}\right)a^2} = b \;\Rightarrow\; S = 2\pi b(b+a(\sin^{-1}c)/c)$

check : when $a \to b$, the ellipse becomes a sphere of radius b; in that case we have $c \to 0$, and it can be shown that $\sin^{-1}(c)/c \to 1$, so $S \to 4\pi b^2$, which is the area of a sphere of radius b

---

8. $\bar{x} = \dfrac{M}{m} = \dfrac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3} = \dfrac{2\cdot(-10)+3\cdot 6+1\cdot x_3}{2+3+1} = \dfrac{x_3-2}{6} = 0 \Rightarrow x_3 = 2$

---

9. $\bar{x} = \dfrac{\int_a^b x(f(x)-g(x))dx}{\int_a^b (f(x)-g(x))dx}$, $\bar{y} = \dfrac{1}{2}\dfrac{\int_a^b (f(x)^2-g(x)^2)dx}{\int_a^b (f(x)-g(x))dx}$, in some cases $g(x) = 0$

- -

9a) $a = 0,\ b = 1,\ g(x) = 0,\ f(x) = x$
$\bar{x} = \frac{\int_0^1 x^2dx}{\int_0^1 xdx} = \frac{2}{3}$ and $\bar{y} = \frac{1}{2}\frac{\int_0^1 x^2dx}{\int_0^1 xdx} = \frac{1}{3}$.
Note that the region is symmetric about line $y = 1-x$, thus $(\bar{x},\bar{y})$ lies on $y = 1-x$. If you notice this, once $\bar{x}$ is known, then $\bar{y}$ can be found without extra calculation.

- -

9b) $a = 0,\ b = 2,\ g(x) = 0,\ f(x) = \sqrt{x(2-x)}$
First consider the curve $y = \sqrt{x(2-x)}$.
Then $y^2 = x(2-x) = 2x-x^2 = -(x^2-2x) = -(x^2-2x+1-1) = -(x-1)^2+1 \Rightarrow (x-1)^2+y^2 = 1$, hence the curve is a circle centered at $(x,y) = (1,0)$ with radius 1. By symmetry we have $\bar{x} = 1$ and then we compute $\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_0^2(\sqrt{x(2-x)})^2dx}{\int_0^2\sqrt{x(2-x)}} = \frac{\frac{1}{2}\int_0^2 x(2-x)dx}{\frac{\pi}{2}} = \frac{2}{\pi}\frac{1}{2}(x^2-\frac{1}{3}x^3)|_0^2 = \frac{1}{\pi}(4-\frac{8}{3}) = \frac{4}{3\pi}$.
(The denominator is the area of a half circle). $\bar{x} = 1$ and $\bar{y} = \frac{4}{3\pi}$

- -

9c) $a = -1,\ b = 1,\ g(x) = 0,\ f(x) = \cosh x$, note that $\cosh x$ is an even function, so the region is symmetric about $x = 0$, and the CM lies on the y-axis $\Rightarrow \bar{x} = 0$, so we just need to calculate $\bar{y}$
first note that $\cosh^2 x = \frac{1}{2}+\frac{1}{2}\cosh 2x$, so $\int\cosh^2 x dx = \int\left(\frac{1}{2}+\frac{1}{2}\cosh 2x\right)dx = \frac{1}{2}x+\frac{1}{4}\sinh 2x$

$$\bar{y} = \frac{\int_{-1}^{1} \frac{1}{2}\cosh^2 x dx}{\int_{-1}^{1} \cosh x dx} = \frac{\int_0^1 \cosh^2 x dx}{2\int_0^1 \cosh x dx} = \frac{\left(\frac{1}{2}x + \frac{1}{4}\sinh 2x\right)\big|_0^1}{2\sinh x|_0^1} = \frac{\frac{1}{2}+\frac{1}{4}\sinh 2}{2\sinh 1} = \frac{\frac{1}{2}+\frac{1}{4}\cdot 2\sinh 1\cosh 1}{2\sinh 1} = \frac{1}{4}\frac{1+\sinh 1\cosh 1}{\sinh 1} \approx 0.6$$

9d) $a = -2$, $b = 2$, $f(x) = \begin{cases} \sqrt{4-x^2} & , \ -2 \le x \le 2 \\ 0 & , \ \text{otherwise} \end{cases}$, $g(x) = \begin{cases} \sqrt{1-x^2} & , \ -1 \le x \le 1 \\ 0 & , \ \text{otherwise} \end{cases}$

The region lies above the x-axis, between the circle of radius 1 and the circle of radius 2. It is symmetric about the y-axis, so $\bar{x} = 0$. We need to compute $\bar{y} = M_x/m$.

$$M_x = \frac{1}{2}\int_a^b (f(x)^2 - g(x)^2)dx = \frac{1}{2}\int_{-2}^{2}(f(x)^2 - g(x)^2)dx = \int_0^2 (f(x)^2 - g(x)^2)dx \text{ (because } f, g \text{ are even)}$$

$$= \int_0^2 f(x)^2 dx - \int_0^2 g(x)^2 dx = \int_0^2 (4-x^2)dx - \int_0^1 (1-x^2)dx = (4x - \tfrac{1}{3}x^3)\Big|_0^2 - (x - \tfrac{1}{3}x^3)\Big|_0^1$$

$$= (8 - \tfrac{8}{3}) - (1 - \tfrac{1}{3}) = \tfrac{14}{3}$$

Since it is a circular region, the area m can be found directly. $m = \frac{1}{2}\cdot\pi\cdot 2^2 - \frac{1}{2}\cdot\pi\cdot 1^2 = \frac{3\pi}{2}$

Then $\bar{y} = \frac{14}{3}\cdot\frac{2}{3\pi} = \frac{28}{9\pi} = 0.9903$, so the CM lies outside the region.

9e) $a = 1$, $b = 2$, $g(x) = 0$, $f(x) = x^{-1}$

$$\bar{x} = \frac{\int_1^2 x\cdot x^{-1}dx}{\int_1^2 x^{-1}dx} = \frac{1}{\ln x|_1^2} = \frac{1}{\ln 2} \text{ and } \bar{y} = \frac{1}{2}\frac{\int_1^2 (x^{-1})^2 dx}{\int_1^2 x^{-1}dx} = \frac{1}{2}\frac{-x^{-1}|_1^2}{\ln 2} = \frac{1}{4\ln 2}$$

---

10. To show that $f(x)$ is minimized when $x = \bar{x}$, we need to show that $f'(\bar{x}) = 0, f''(\bar{x}) > 0$.

$$f(x) = \sum_{i=1}^{n} m_i(x - x_i)^2 \Rightarrow f'(x) = \sum_{i=1}^{n} 2m_i(x - x_i) = \sum_{i=1}^{n} 2m_i x - \sum_{i=1}^{n} 2m_i x_i = 2x\sum_{i=1}^{n} m_i - 2\sum_{i=1}^{n} m_i x_i$$

$$= 2xm - 2M \ , \text{ where } m = \sum_{i=1}^{n} m_i \ , \ M = \sum_{i=1}^{n} m_i x_i$$

then $f'(x) = 0 \Rightarrow 2xm - 2M = 0 \Rightarrow x = \frac{M}{m} = \bar{x}$ and $f''(x) = 2m > 0$ for all x

---

11. prob$(50 \le X \le 60)$ = prob$(\mu - \sigma \le X \le \mu + \sigma) \approx 0.68 = 68\%$ (see page 37 of the lecture notes, this assumes we know erf$(1/\sqrt{2}) = 0.6827$, this must be computed using a calculator or Maple or other such system, any such values needed on the exam will be given)

12a) $\int_{-\infty}^{\infty} f(t)dt = \int_0^5 \frac{6}{625}t^2(5-t)^2 dt = \frac{6}{625}\int_0^5 t^2(25 - 10t + t^2)dt = \frac{6}{625}\int_0^5 (25t^2 - 10t^3 + t^4)dt = \frac{6}{625}\left(\frac{25}{3}t^3 - \frac{10}{4}t^4 + \frac{t^5}{5}\right)\Big|_0^5 = 1$ and $f(t) \ge 0$, thus $f(t)$ defines a valid pdf

mean : $\mu = \int_0^5 tf(t)dt = \frac{6}{625}\int_0^5 (25t^3 - 10t^4 + t^5)dt = \cdots = \frac{5}{2}$

note : $f(t)$ is symmetric about $t = \frac{5}{2}$ (show it); this also implies that $\mu = \frac{5}{2}$

12b) $1000 \cdot \text{prob}(T \ge 3) = 1000 \cdot \int_3^5 f(t)dt = 1000 \cdot \int_3^5 \frac{6}{625}t^2(5-t)^2 dt = \cdots \approx 317$ batteries

---

13. $f(t) = ce^{-ct}$ where $c = \frac{1}{20}$, t is measured in minutes, $\mu = \frac{1}{c}$, thus $c = \frac{1}{\mu}$.

a) prob$(X \le 10) = \int_0^{10} \frac{1}{20}e^{-\frac{t}{20}}dt = 1 - \frac{1}{\sqrt{e}} \approx 0.375$

b) prob$(X \ge 30) = \int_{30}^{\infty} \frac{1}{20}e^{-\frac{t}{20}}dt = e^{-\frac{3}{2}} \approx 0.24$

c) Solve $\frac{1}{2} = \int_m^{\infty} f(t)dt \Rightarrow \frac{1}{2} = e^{-\frac{m}{20}} \Rightarrow m = 20\ln 2 \approx 20 \times 0.7 = 14$ minutes. (The median $m = \mu\ln 2 < \mu$ for exponential distribution.)

---

14. prob$(\mu - 2\sigma < X < \mu + 2\sigma) = \int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}\,dx \quad \left(\text{set } y = (x-\mu)/\sqrt{2}\sigma, dy = dx/\sqrt{2}\sigma\right)$

$$= \frac{1}{\sqrt{\pi}}\int_{-\sqrt{2}}^{\sqrt{2}} e^{-y^2}dy = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{2}} e^{-y^2}dy = \text{erf}(\sqrt{2}) = 0.9545 \approx 95\%$$

Hence for a normal distribution, 95% of the values are within two standard deviations of the mean.

---

15. The exponential pdf is $f(t) = \begin{cases} ce^{-ct} \ , & t \ge 0 \\ 0 \ , & t < 0 \end{cases}$ with mean $\mu = 1/c$.

$\sigma^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$

$= \int_0^\infty (x-\mu)^2 ce^{-cx}\,dx \quad \big(\text{set } u=(x-\mu)^2\,,\ dv = ce^{-cx}\,dx \Rightarrow du = 2(x-\mu)dx\,,\ v = -e^{-cx}\big)$

$= -(x-\mu)^2 e^{-cx}\Big|_0^\infty + 2\int_0^\infty (x-\mu)e^{-cx}\,dx$

$= \mu^2 + 2\int_0^\infty (x-\mu)e^{-cx}\,dx \quad \big(\text{set } u = x-\mu\,,\ dv = e^{-cx}\,dx \Rightarrow du = dx\,,\ v = -\frac{1}{c}e^{-cx}\big)$

$= \mu^2 - 2\frac{x-\mu}{c}e^{-cx}\Big|_0^\infty + \frac{2}{c}\int_0^\infty e^{-cx}\,dx = \mu^2 - \frac{2\mu}{c} - \frac{2}{c^2}e^{-cx}\Big|_0^\infty = \mu^2 - \frac{2\mu}{c} + \frac{2}{c^2} = \mu^2,$

where in the last step we substituted $c = 1/\mu$. Hence $\sigma = \mu$.

---

16. Functions (a), (b), (c), (e) are solutions of the differential equation; functions (d), (f) are not.

---

17. a) $y = 1$: unstable
17. b) $y = -1$: stable , $y = 1$: unstable
17. c) $y = 1$: one side stable and one side unstable
17. d) $y = 1$: stable , $y = 2$: unstable
17. e) $y = 2k\pi$ (k is integer) : unstable , $y = (2k+1)\pi$: stable

---

18. $E(t) = \frac{1}{2}mv^2 + V(x) \Rightarrow E(t) = \frac{1}{2}mx'(t)^2 + V(x(t))$
$\Rightarrow E'(t) = \frac{1}{2}m \cdot 2x'(t)x''(t) + V'(x(t))x'(t) = x'(t)[mx''(t) + V'(x(t))] = x'(t)[f(x(t)) - f(x(t))] = 0$
$\Rightarrow E'(t) = 0 \Rightarrow E(t) = \text{constant}$
Hence if the kinetic energy increases, then the potential energy decreases.

19a) $y' = y \Rightarrow \frac{dy}{dt} = y \Rightarrow \frac{dy}{y} = dt \Rightarrow \ln y = t + c \Rightarrow y = e^c \cdot e^t$, $e^c = 1$ using $y(0) = 1$
The solution is $y = y(0)e^t \Rightarrow y = e^t$.

- -

19b) $y' = ty \Rightarrow \frac{dy}{dt} = ty \Rightarrow \frac{dy}{y} = tdt \Rightarrow \ln y = \frac{1}{2}t^2 + c \Rightarrow y = e^c \cdot e^{\frac{t^2}{2}}$, $e^c = 1$ using $y(0) = 1$
The solution is $y = y(0)e^{\frac{t^2}{2}} \Rightarrow y = e^{\frac{t^2}{2}}$.

- -

19c) $y' = y^2 \Rightarrow \frac{dy}{dt} = y^2 \Rightarrow \frac{dy}{y^2} = dt \Rightarrow -\frac{1}{y} = t + c \Rightarrow y = \frac{1}{-t-c}$, $c = -1$ using $y(0) = 1$
The solution is $y = \frac{1}{1-t}$. The graph has a vertical asymptote at $t = 1$.

- -

19d) $y' = y(1-y) \Rightarrow \frac{dy}{dt} = y(1-y) \Rightarrow \frac{dy}{y(1-y)} = dt \Rightarrow \frac{dy}{y} + \frac{dy}{1-y} = dt \Rightarrow \ln y - \ln(1-y) = t + c \Rightarrow \ln\frac{y}{(1-y)} = t + c \Rightarrow \frac{y}{(1-y)} = e^t \cdot e^c \ldots$
One can find the general solution $y(t)$, but the initial condition $y(0) = 1$ is a constant solution, so the solution will stay at $y = 1$ for all $t \geq 0$, so the solution in this case is $y(t) = 1$, and the general solution is not needed.

---

20. $y(t)$ = amount of salt (kg) in the tank at time t (min)
$y'(t)$ = rate of change of amount = (rate coming in) $-$ (rate going out)

rate coming in $= 0.05\dfrac{\text{kg}}{\text{L}} \cdot 5\dfrac{\text{L}}{\text{min}} = 0.25\dfrac{\text{kg}}{\text{min}}$, rate going out $= \dfrac{y\,\text{kg}}{1000\,\text{L}} \cdot 5\dfrac{\text{L}}{\text{min}} = 0.005y\dfrac{\text{kg}}{\text{min}}$

$y' = 0.25 - 0.005 \Rightarrow y' = -0.005(y - 50)$

Note that we have written the equation in the form $y' = k(y - T)$, which is Newton's law of cooling/heating, with $k = -0.005, T = 50$. The solution is $y(t) = T + (y_0 - T)e^{kt} = 50 + (0 - 50)e^{-0.005t} = 50(1 - e^{-0.005t})$, using the initial condition $y(0) = y_0 = 0$, since the tank initially contains pure water.

a) After one hour = 60 minutes we have $y(60) = 50(1 - e^{-0.005\cdot 60}) = 50(1 - e^{-0.3}) \approx 13$ kg.

b) $\lim\limits_{t\to\infty} y(t) = \lim\limits_{t\to\infty} 50(1 - e^{-0.005t}) = 50$ kg
Note that this is the quantity of salt present in the tank if the tank contained only sea water.

21a) $x(t)$ = value of new-type bills (in units of \$1B) in circulation at time t (days)
$x'(t)$ = rate of change of new-type bills in circulation = (rate going in) - (rate going out)
rate going in $= 50\frac{\text{M}}{\text{day}} = 0.05\frac{\text{B}}{\text{day}}$, rate going out $= 0.05\frac{\text{B}}{\text{day}} \cdot \frac{x}{10} \Rightarrow x' = 0.05 - 0.005x$

- -

21b) The equation can be written as $x' = -0.005(x - 10)$, which is the same as Newton's law of cooling/heating, $x' = k(x - T)$, with $k = -0.005$ and $T = 10$. Hence $x(t) = T + (x_0 - T)e^{-kt} = 10 + (0 - 10)e^{-0.005t} = 10(1 - e^{-0.005t})$, where we used the initial condition $x_0 = x(0) = 0$.

- -

21c) 90% means $x(t) = 9 \Rightarrow 10(1 - e^{-0.005t}) = 9 \Rightarrow 1 - e^{-0.005t} = 0.9 \Rightarrow -e^{-0.005t} = -0.1 \Rightarrow e^{-0.005t} = 0.1 \Rightarrow -0.005t = \ln(0.1) = -\ln 10 \Rightarrow t = 200 \ln 10 \approx 200 \cdot 2.3 = 460$ days

---

22. $c' = k(a - c)(b - c)$, where $k > 0$, $a = 1$, $b = 2 \Rightarrow c' = k(1 - c)(2 - c)$, it has two constant solutions $c = 1$ and $c = 2$ (ie., let $c' = 0$, solve the equation), $c = 1$ is stable while $c = 2$ is unstable, since for $0 \le c < 1$, $c' > 0$, $c(t)$ increases (towards $c = 1$), for $1 < c < 2$, $c' < 0$, $c(t)$ decreases (towards $c = 1$), for $c > 2$, $c' > 0$, $c(t)$ increases (away from $c = 2$).

To find the product concentration, we solve $c' = k(1 - c)(2 - c)$ by separation of variables, $\frac{dc}{(1-c)(2-c)} = kdt$. Using partial fractions, we write $\frac{1}{(1-c)(2-c)} = \frac{1}{1-c} - \frac{1}{2-c}$, and integrate to obtain $\ln\left|\frac{2-c}{1-c}\right| = kt + d$, or $\frac{2-c}{1-c} = De^{kt}$, for some constant D (using the initial condition $c(0) = 0$, we find this constant to be $D = 2$). We now solve the equation for $c(t)$, writing $2 - c = (1 - c)De^{kt}$. Rearranging terms and simplifying, we get $c(t) = 1 - \frac{1}{2e^{kt}-1}$.

Given that $c(0) = 0$ at $t = 0$, the asymptotic value of $c(t)$ is $c = 1$ as $t \to \infty$. The asymptotic value will not change if $c(0) = 1.5$ mole/L. This can also be seen using the phase plane.

---

23. $mc_pT' = -e\sigma T^4 \Rightarrow \frac{T'}{T^4} = -\frac{e\sigma}{mc_p} \Rightarrow \int \frac{T'}{T^4}dt = \int -\frac{e\sigma}{mc_p}dt \Rightarrow -\frac{1}{3T^3} = -\frac{e\sigma}{mc_p}t + c$
initial condition : $T(0) = T_0 \Rightarrow c = -\frac{1}{3T_0^3} \Rightarrow -\frac{1}{3T^3} = -\frac{e\sigma}{mc_p}t - \frac{1}{3T_0^3} \Rightarrow T(t) = \left(3\frac{e\sigma}{mc_p}t + \frac{1}{T_0^3}\right)^{-1/3}$
In the limit $t \to \infty$, $T(t) \sim t^{-1/3}$, so $\alpha = \frac{1}{3}$.

24a) The decay rate of radioactive sample satisfies $y' = -ky$. By separation of variables, we find the solution $y(t) = y_0e^{-kt}$ where y_0 is initial mass at $t = 0$. It is given that $y(2) = 128$ after 2 hours and $y(5) = 2$ after 5 hours, $\Rightarrow \begin{cases} 128 = y_0e^{-2k} \\ 2 = y_0e^{-5k} \end{cases}$. Dividing the first equation by the second equation yields $64 = e^{3k} \Rightarrow e^k = 64^{1/3} = 4$. Substituting in the first equation gives $128 = y_04^{-2} \Rightarrow y_0 = 128 \cdot 16 = 2048$.
24b) Let $y(t) = 1$ kg. Then $1 = 2048 \cdot 4^{-t} \Rightarrow 4^t = 2048 \Rightarrow t = \frac{\ln 2048}{\ln 4} = \frac{\ln 2^{11}}{\ln 2^2} = \frac{11}{2} \Rightarrow t = 5.5$ hours.

---

25. The initial value of the investment is $x = \$500$ and the annual interest rate is $r = 12\% = 0.12$.

- -

a) If the interest is compounded annually, then the value of the investment after 10 years is $x(1 + r)^n = 500(1.12)^{10} \approx \$1552.92$.

- -

b) If the interest is compounded continuously, then the value of the investment after 10 years is $xe^{rt} = 500e^{0.12(10)} \approx \$1660.06$.

- -

c) The equivalent annual interest rate r_{eq} is found by solving the equation $e^r = 1 + r_{eq}$. Since $e^r = e^{0.12} \approx 1.127 = 1 + 0.127$, the equivalent annual interest rate is $r_{eq} = 0.127$. In other words, 12% compounded continuously is the same as 12.7% compounded annually.

---

26. Newton's Law of cooling/heating in the form $y' = k(y - T)$ implies $y(t) = T + (y_0 - T)e^{kt}$, where $y_0 = 21$ at $t = 0$ is the initial temperature.

We know $y(1) = T + (21-T)e^k = 27$ and $y(2) = T + (21-T)e^{2k} = 30$.
The 1st equation gives $e^k = \frac{27-T}{21-T}$ and the 2nd equation gives $e^{2k} = \frac{30-T}{21-T} = \left(\frac{27-T}{21-T}\right)^2$.
$\Rightarrow (30-T)(21-T)^2 = (21-T)(27-T)^2 \Rightarrow (30-T)(21-T) = (27-T)^2$
$\Rightarrow 630 - 51T + T^2 = 729 - 54T + T^2 \Rightarrow 3T = 729 - 630 = 99 \Rightarrow T = 33°\text{C}$

27a) The correct equation is (iii) $y' = r - ky$.
27b) $r - ky = 0 \Rightarrow m = \frac{r}{k} = C$ (phase plane, done in class, stable)
27c) The equation in a) can be rewritten as $y' = -k(y - \frac{r}{k})$. This is the same as Newton's Law of cooling/heating and the solution is $y(t) = \frac{r}{k} + (y_0 - \frac{r}{k})e^{-kt}$.
The endowment dropped to $\frac{1}{2}C$ last year, so we take $y_0 = \frac{1}{2}C$. The solution is
$y(t) = \frac{r}{k} + (\frac{1}{2}C - \frac{r}{k})e^{-kt} = C - \frac{1}{2}Ce^{-kt}$.
To find how long it will take to recover to $\frac{3}{4}C$, we set
$y(t) = \frac{3}{4}C \Rightarrow C - \frac{1}{2}Ce^{-kt} = \frac{3}{4}C \Rightarrow -\frac{1}{2}Ce^{-kt} = -\frac{1}{4}C \Rightarrow e^{-kt} = \frac{1}{2} \Rightarrow t = -\frac{1}{k}\ln\frac{1}{2} = \frac{1}{k}\ln 2$.
For $k = \frac{1}{20}\,\text{year}^{-1}$ and $\ln 2 \approx 0.7$, we have $t \approx 14\,$years.

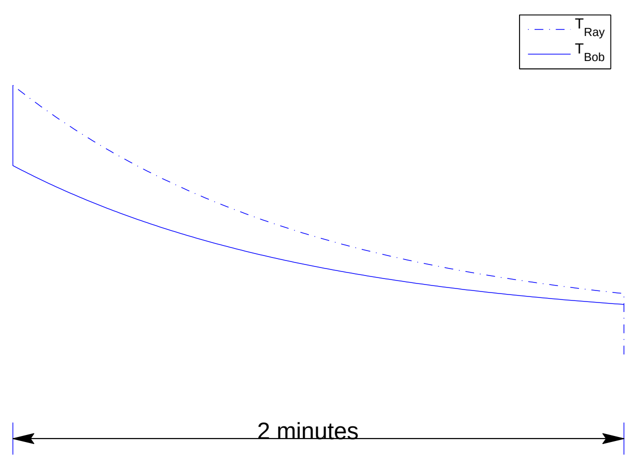
---

28. Newton's law of cooling/heating : $y' = k(y-T) \Rightarrow y(t) = T + (y_0 - T)e^{kt}$, $k < 0$
Bob : adds milk, then waits 2 mins , Ray : waits 2 mins, then adds milk

initial condition : $y_{\text{Bob}}(0) = \dfrac{8T_h + T_c}{9}$, $y_{\text{Ray}}(0) = T_h$

$$y_{\text{Bob}}(2) - y_{\text{Ray}}(2) = \left(T_a + \left(\frac{8T_h + T_c}{9} - T_a\right)e^{2k}\right) - \left(\frac{8(T_a + (T_h - T_a)e^{2k}) + T_c}{9}\right)$$
$$= T_a - \frac{8}{9}T_a - \frac{1}{9}T_c + \left(\frac{8T_h + T_c}{9} - T_a - \frac{8}{9}(T_h - T_a)\right)e^{2k}$$
$$= \frac{1}{9}(T_a - T_c) + \frac{1}{9}(T_c - T_a)e^{2k} = \frac{1}{9}(T_a - T_c)(1 - e^{2k}) > 0 \Rightarrow y_{\text{Bob}}(2) > y_{\text{Ray}}(2)$$

Ray ends up with the colder coffee; this can be understood as follows. According to Newton's law, the larger the temperature difference between the object and its surroundings, the faster the temperature changes. Since Bob reduced the initial temperature difference, he reduced the temperature rate of change, so his coffee cools more slowly during the 2 minutes. See the plots of $y_{\text{Bob}}(t), y_{\text{Ray}}(t)$.



---

29. Let $y(t)$ be the fraction of the population who have already heard the rumor at time t. Then $1 - y(t)$ is the fraction of the population who have not already heard the rumor at time t.

The rate of change of the fraction of the population who have heard the rumor is $y' = ky(1-y)$, for some $k > 0$. This makes sense because if $y = 0$ or $y = 1$, then $y' = 0$.

This is a logistic equation with general solution $y(t) = \frac{y_0}{y_0 + (1-y_0)e^{-kt}}$. Set $t = 0$ at 8am, then $y_0 = \frac{10}{1000} = 0.01$. Then at 9am, $t = 1$ hour, $y(1) = \frac{20}{1000} = 0.02$. Substituting these into the solution yields $0.02 = \frac{0.01}{0.01 + 0.99e^{-k}} \Rightarrow 0.01 + 0.99e^{-k} = \frac{1}{2} \Rightarrow e^{-k} = \frac{49}{99}$. The solution becomes $y(t) = \frac{0.01}{0.01 + 0.99(\frac{49}{99})^t}$; note that $e^{-kt} = (e^{-k})^t$. Then $y(t) = \frac{1}{2} \Rightarrow \frac{0.01}{0.01 + 0.99(\frac{49}{99})^t} = \frac{1}{2} \Rightarrow 0.01 + 0.99(\frac{49}{99})^t = 0.02 \Rightarrow (\frac{49}{99})^t = \frac{1}{99} \Rightarrow t = \ln(99)/(\ln(99) - \ln(49)) = 6.5337$, i.e. around 2:30pm.

30a) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=\sum\limits_{n=0}^{\infty}\left(\frac{1}{2}\right)^n=\frac{1}{1-(\frac{1}{2})}=2$

30b) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}-\cdots=\sum\limits_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n=\frac{1}{1-(-\frac{1}{2})}=\frac{2}{3}$

30c) $1+\frac{1}{2}-\frac{1}{4}+\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\cdots=(1+\frac{1}{2}-\frac{1}{4})+(\frac{1}{8}+\frac{1}{16}-\frac{1}{32})+(\frac{1}{64}+\frac{1}{128}-\frac{1}{256})+\cdots$

$=\frac{5}{4}+\frac{5}{32}+\frac{5}{256}+\cdots=\frac{5}{4}(1+\frac{1}{8}+\frac{1}{64}+\cdots)=\frac{5}{4}\sum\limits_{n=0}^{\infty}(\frac{1}{8})^n=\frac{5}{4}\cdot\frac{1}{1-\frac{1}{8}}=\frac{5}{4}\cdot\frac{8}{7}=\frac{10}{7}$

---

31. (a) $\sum\limits_{n=0}^{\infty}(2x-1)^n$ is a geometric series with $r=2x-1$; hence the series converges if and only if $-1<r<1 \Leftrightarrow -1<2x-1<1 \Leftrightarrow 0<2x<2 \Leftrightarrow 0<x<1$.

(b) If $0<x<1$, then we have $\sum\limits_{n=0}^{\infty}(2x-1)^n=\sum\limits_{n=0}^{\infty}r^n=1+r+r^2+r^3+\cdots=\frac{1}{1-r}=\frac{1}{1-(2x-1)}=\frac{1}{2-2x}$.

---

32. Determine whether the series converges or diverges.

a) $\sum\limits_{n=1}^{\infty}\frac{n}{n+1}$: diverges, $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{n}{n+1}=1$, a necessary condition for convergence is $\lim\limits_{n\to\infty}a_n=0$

- -

b) $\sum\limits_{n=1}^{\infty}\frac{n}{n^2+1}$: diverges by integral test, $\int_1^{\infty}f(x)dx=\int_1^{\infty}\frac{x}{x^2+1}dx=\frac{1}{2}\ln(x^2+1)\Big|_1^{\infty}=\infty-\frac{1}{2}\ln 2=\infty$

- -

c) $\sum\limits_{n=1}^{\infty}\frac{1}{n^2+1}$: conerges by comparison test, $\sum\limits_{n=1}^{\infty}\frac{1}{n^2+1}\le\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$: converges by p-test

---

33a) $\int\cosh 2x dx=\frac{1}{2}\sinh 2x$

33b) $\int\tanh 2x dx=\frac{1}{2}\int\tanh(2x)d(2x)=\frac{1}{2}\int\frac{\sinh 2x}{\cosh 2x}d(2x)=\frac{1}{2}\int\frac{1}{\cosh 2x}d\cosh 2x=\frac{1}{2}\ln(\cosh 2x)$

33c) $\int\cosh^2 x dx=\int(\frac{e^x+e^{-x}}{2})^2dx=\int\frac{e^{2x}+2+e^{-2x}}{4}dx=\frac{1}{4}\left(\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}\right)=\frac{1}{4}\sinh 2x+\frac{1}{2}x$

---

34. $\sinh(x)=\dfrac{e^x-e^{-x}}{2}$, $\cosh(x)=\dfrac{e^x+e^{-x}}{2}$

a) LHS $=\sinh(x+y)=\dfrac{e^{x+y}-e^{-x-y}}{2}$

RHS $=\sinh x\cosh y+\cosh x\sinh y=\dfrac{e^x-e^{-x}}{2}\cdot\dfrac{e^y+e^{-y}}{2}+\dfrac{e^x+e^{-x}}{2}\cdot\dfrac{e^y-e^{-y}}{2}$

$=\dfrac{e^{x+y}+e^{x-y}-e^{-x+y}-e^{-x-y}}{4}+\dfrac{e^{x+y}-e^{x-y}+e^{-x+y}-e^{-x-y}}{4}=\dfrac{2e^{x+y}-2e^{-x-y}}{4}=$ LHS

- -

b) One can prove it by the definitions of sinh, cosh, as in (a), or by using $\sinh(-y)=-\sinh y,\cosh(-y)=\cosh y$ (i.e. sinh is an odd function, cosh is an even function) and based on a):

$\sinh(x-y)=\sinh[x+(-y)]\overset{a)}{=}\sinh(x)\cosh(-y)+\cosh(x)\sinh(-y)=\sinh x\cosh y-\cosh x\sinh y$

Note : For parts c) and d), one can prove them by the definitions of $\sinh,\cosh$ similar to a), or by using the relationships $(\sinh x)'=\cosh x,(\cosh x)'=\sinh x$ and the identity a) (see below).

- -

c) Taking derivative for both sides of a) with respect to x, regarding y as a constant, we have:

$\cosh(x+y)=[\sinh(x+y)]'=[\sinh x\cosh y+\cosh x\sinh y]'=(\sinh x)'\cosh y+(\cosh x)'\sinh y$
$=\cosh x\cosh y+\sinh x\sinh y$

d) Differentiating both sides of b) with respect to x, regarding y as constant yields the result.

---

35. a) $\lim\limits_{x\to\pm\infty}\operatorname{sech} x=0$ b) $\frac{d}{dx}\operatorname{sech} x=\frac{d}{dx}\frac{1}{\cosh x}=\frac{-\sinh x}{\cosh^2 x}=-\tanh x\operatorname{sech} x$ c) like a bell curve

---

36. $\lim_{x\to\infty} \dfrac{\sinh x}{\cosh x} = \lim_{x\to\infty} \dfrac{\frac{e^x+e^{-x}}{2}}{\frac{x^2-e^{-x}}{2}} = \lim_{x\to\infty} \dfrac{e^x+e^{-x}}{e^x-e^{-x}} = \lim_{x\to\infty} \dfrac{1+e^{-2x}}{1-e^{-2x}} = 1 \Rightarrow \sinh x \sim \cosh x$ as $x \to \infty$

---

37. a) $f(x) = x^{-1} \Rightarrow f'(x) = -x^{-2}, f''(x) = 2x^{-3}$
$a = 1 \Rightarrow f(1) = 1,\ f'(1) = -1, f''(1) = 2$
$T_1(x) = f(a) + f'(a)(x-a) \Rightarrow T_1(x) = 1 - (x-1) = 2 - x$

- -

b) $f(x) = \sin x \Rightarrow f'(x) = \cos x, f''(x) = -\sin x$
$a = \frac{\pi}{2} \Rightarrow f(\frac{\pi}{2}) = 1,\ f'(\frac{\pi}{2}) = 0, f''(\frac{\pi}{2}) = -1$
$T_1(x) = f(a) + f'(a)(x-a) \Rightarrow T_1(x) = 1 - 0(x - \frac{\pi}{2}) = 1$

- -

c) $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}, f''(x) = -\frac{1}{4\sqrt{x^3}}$
$a = 4 \Rightarrow f(4) = \sqrt{4} = 2, f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}, f''(4) = -\frac{1}{4\sqrt{4^3}} = -\frac{1}{32}$
$T_1(x) = f(a) + f'(a)(x-a) \Rightarrow T_1(x) = 2 + \frac{1}{4}(x-4)$
approximation of $\sqrt{5} = 2.2361$ (viewing $\sqrt{5}$ as the value of $\sqrt{x}$ at $x = 5$)
$T_1(5) = 2 + \frac{1}{4}(5-4) = \frac{9}{4} = 2.25$, the error is $|f(5) - T_1(5)| = |2.2361 - 2.25| = 0.0139$

---

38. Let $L = \lim_{n\to\infty} n((1+\dfrac{1}{n})^n - e)$. This limit L has the form $\infty \cdot 0$, so to apply l'Hôpital's rule we move n to the denominator, $L = \lim_{n\to\infty} \dfrac{(1+\frac{1}{n})^n - e}{\frac{1}{n}}$. Now we have a limit of the type $\frac{0}{0}$, for which we can apply l'Hôpital's rule. However it is more convenient to substitute $u = \frac{1}{n}$.

$$L = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^n - e}{\frac{1}{n}} = \lim_{u\to 0} \frac{(1+u)^{1/u} - e}{u}$$

Now differentiate the numerator, but note that $(1+u)^{1/u} = e^{\ln((1+u)^{1/u})} = e^{\frac{\ln(1+u)}{u}}$.

$$\frac{d}{du}(1+u)^{1/u} = e^{\frac{\ln(1+u)}{u}} \cdot \frac{u \cdot \frac{1}{1+u} - \ln(1+u)}{u^2}$$

$$\Rightarrow L = \lim_{u\to 0} \frac{(1+u)^{1/u} - e}{u} = \lim_{u\to 0} e^{\frac{\ln(1+u)}{u}} \cdot \frac{u \cdot \frac{1}{1+u} - \ln(1+u)}{u^2} = \lim_{u\to 0} e^{\frac{\ln(1+u)}{u}} \cdot \lim_{u\to 0} \frac{u \cdot \frac{1}{1+u} - \ln(1+u)}{u^2}$$

$$= \lim_{u\to 0}(1+u)^{1/u} \cdot \lim_{u\to 0} \frac{u \cdot \frac{1}{1+u} - \ln(1+u)}{u^2} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \cdot \lim_{u\to 0} \frac{u - (1+u)\ln(1+u)}{(1+u)u^2}$$

$$= e \cdot \lim_{u\to 0} \frac{1 - \left[\frac{1+u}{1+u} + \ln(1+u)\right]}{(1+u)\cdot 2u + u^2} = e \cdot \lim_{u\to 0} \frac{-\ln(1+u)}{2u + 3u^2} = e \cdot \lim_{u\to 0} \frac{-\frac{1}{1+u}}{2+6u} = e \cdot -\frac{1}{2} = -\frac{e}{2} \quad \underline{\text{ok}}$$

This confirms a result about Euler's method in the Table on page 48 of the notes.

---

39. The <u>heat kernel</u>, defined by $f(x,t) = \dfrac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}$, is used in the study of heat conduction, where $f(x,t)$ is the temperature induced by a point heat source at location x and time $t > 0$.

a) Note that $f(x,t)$ is the pdf of a normally distributed random variable; what are μ and σ?

The pdf of a normal distribution is $\dfrac{1}{\sqrt{2\pi}\sigma}\, e^{-(x-\mu)^2/2\sigma^2}$.

Then the heat kernel $f(x,t)$ corresponds to $\mu = 0, \sigma = \sqrt{2t}$.

b) Show that the heat kernel satisfies the partial differential equation $\dfrac{\partial f}{\partial t} = \dfrac{\partial^2 f}{\partial x^2}$ (the <u>heat equation</u>).

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t} = \frac{1}{\sqrt{4\pi}}\frac{\partial}{\partial t}\left(t^{-1/2} e^{-x^2/4t}\right) = \frac{1}{\sqrt{4\pi}}\left(t^{-1/2}\frac{\partial}{\partial t}\, e^{-x^2/4t} + \frac{\partial}{\partial t}\left(t^{-1/2}\right) e^{-x^2/4t}\right)$$

$$= \frac{1}{\sqrt{4\pi}} \left(t^{-1/2} \cdot e^{-x^2/4t} \cdot \frac{x^2}{4t^2} - \frac{1}{2} t^{-3/2} \cdot e^{-x^2/4t} \right) = \frac{1}{\sqrt{4\pi}} \left(\frac{x^2}{4} t^{-5/2} - \frac{1}{2} t^{-3/2} \right) e^{-x^2/4t}$$

$$= \frac{1}{\sqrt{4\pi}} \frac{1}{2t^{3/2}} \left(\frac{x^2}{2t} - 1 \right) e^{-x^2/4t}$$

- -

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} = \frac{1}{\sqrt{4\pi t}} \frac{\partial}{\partial x} e^{-x^2/4t} = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \cdot \frac{-x}{2t} = \frac{1}{\sqrt{4\pi t}} \frac{-1}{2t} x e^{-x^2/4t}$$

- -

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \frac{1}{\sqrt{4\pi t}} \frac{-1}{2t} x e^{-x^2/4t} = \frac{1}{\sqrt{4\pi t}} \frac{-1}{2t} \frac{\partial}{\partial x} x e^{-x^2/4t}$$

$$= \frac{1}{\sqrt{4\pi t}} \frac{-1}{2t} \left(x \cdot e^{-x^2/4t} \cdot \frac{-x}{2t} + e^{-x^2/4t} \right) = \frac{1}{\sqrt{4\pi t}} \frac{-1}{2t} \left(\frac{-x^2}{2t} + 1 \right) e^{-x^2/4t}$$

$$= \frac{1}{\sqrt{4\pi}} \frac{1}{2t^{3/2}} \left(\frac{x^2}{2t} - 1 \right) e^{-x^2/4t} \qquad \underline{\text{ok}}$$