Math 156 Applied Honors Calculus II Fall 2023

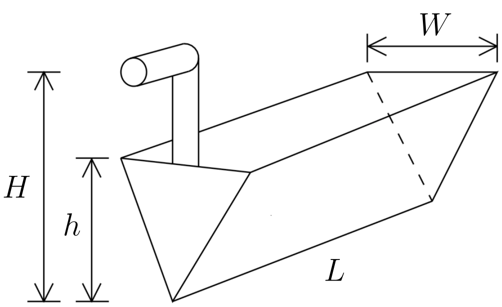
hw6 , due: Friday, October 13 at 4pm

1. Sketch the circle $x^2 + y^2 = r^2$ and the line $y = r$ in the xy-plane. Find the area of the surface obtained by rotating the circle about the line.

2. An observer is located at height h above the North pole of a sphere of radius r. (a) Show that the observer can see a portion of the sphere with area $S = 2\pi r^2 h/(r+h)$. (hint: draw a picture, use the formula for the surface area obtained by rotating a curve about an axis) (b) Check the answer in two limits, $h \to 0$ and $h \to \infty$, and explain why the results make sense.

3. Four point masses $m_1 = 6, m_2 = 5, m_3 = 1, m_4 = 4$ are located in the xy-plane at $(x_1, y_1) = (1, -2), (x_2, y_2) = (3, 4), (x_3, y_3) = (-3, -7), (x_4, y_4) = (6, -1)$. Sketch the mass distribution, find the center of mass, and indicate the CM on the sketch.

4. A tank with the indicated shape is full of water with density ρ. (a) Find the work done in pumping the water to the top of the outlet. Express the answer in terms of ρ, g, h, W, L, H. (A similar problem was on hw3). (b) Suppose the entire water mass is concentrated into a single point mass located at the center of mass of the tank. In class we showed that the CM of an isoceles triangle is located at a distance 1/3 of the triangle height from the base (we showed this for a special case, but you may assume it is true in general). Compute the work done in raising the point mass to the top of the outlet. (c) Show that part (a) and part (b) give the same result.



5. Evaluate the limit. $\lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$

6. This problem introduces the hyperbolic functions.

a) Define $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\sinh x = \dfrac{e^x - e^{-x}}{2}$. Show that $\cosh x$ is an even function (i.e. $f(-x) = f(x)$) and $\sinh x$ is an odd function (i.e. $f(-x) = -f(x)$).

b) Sketch $y = \cosh x$ and $y = \sinh x$ on the same plot for $-\infty < x < \infty$.

c) Show that $\cosh^2 x - \sinh^2 x = 1$.

This shows that the point $(X, Y) = (\cosh x, \sinh x)$ lies on the hyperbola $X^2 - Y^2 = 1$ in the XY-plane, and this is why $\cosh x, \sinh x$ are called hyperbolic trigonometric functions. The familiar functions $\cos x, \sin x$ are sometimes called circular trigonometric functions because they satisfy the equation $\cos^2 x + \sin^2 x = 1$, which implies that the point $(X, Y) = (\cos x, \sin x)$ lies on the circle $X^2 + Y^2 = 1$ in the XY-plane.

d) Find $\frac{d}{dx}\cosh x$, $\frac{d}{dx}\sinh x$.

e) Define $\tanh x = \dfrac{\sinh x}{\cosh x}$. Show that $\tanh x$ is an odd function. Find $\frac{d}{dx}\tanh x$.

f) Evaluate $\lim\limits_{x\to\infty}\tanh x$, $\lim\limits_{x\to-\infty}\tanh x$. Sketch the graph of $\tanh x$ for $-\infty < x < \infty$.

A final comment - by analogy with $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\sinh x = \dfrac{e^x - e^{-x}}{2}$, it can be shown that $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$, $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$, where $i = \sqrt{-1}$; this is a consequence of Euler's formula; we'll see this later in the course.