

hw7 , due: Tuesday, October 24 at 4pm

1. Consider the region between the curves $y = x^m$ and $y = x^n$, for $0 \leq x \leq 1$, where m, n are integers with $0 \leq m < n$. (a) Sketch the region and label the curves for general m and n . (b) Find the center of mass of the region (\bar{x}, \bar{y}) in terms of m and n . (c) Consider the case $n = m + 1$. Make a table as follows, column 1: m , column 2: n , column 3: \bar{x} , column 4: \bar{x}^n , column 5: \bar{y} , column 6: \bar{x}^m , column 7: CM lies in region? (yes or no). Take $m = 0, 1, 2, 3$. In columns 3-6, give the value in fractional form and decimal form with 4 decimal digits (e.g. $\frac{1}{3} = 0.3333$). Show that the CM lies inside the region for $m = 0, 1, 2$ and outside for $m = 3$.

2. Let X be a random variable with pdf $f(x) = \frac{3}{64}x\sqrt{16 - x^2}$ for $0 \leq x \leq 4$ and $f(x) = 0$ for all other x . (a) Sketch the graph of $f(x)$. (b) Verify that $f(x)$ is a valid pdf. (c) Find $\text{prob}(0 \leq X \leq 2)$.

3. The pdf of a normal distribution is $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x - \mu)^2/2\sigma^2}$ with constants $\mu, \sigma > 0$.

Show that (a) $\int_{-\infty}^{\infty} f(x)dx = 1$, (b) $\int_{-\infty}^{\infty} xf(x)dx = \mu$, (c) $\int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx = \sigma^2$.

This verifies that μ is the mean and σ is the standard deviation.

(hint: substitute $t = (x - \mu)/\sqrt{2}\sigma$, use the fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$)

4. In class we considered the waiting time in the supermarket checkout line as a random variable T with exponential pdf $f(t)$. Let the average waiting time be $\mu = 5$ minutes and show that the median waiting time is $m = 3.5$ minutes. Explain how it is possible for the average waiting time to be longer than the median waiting time.

5. Use separation of variables to find the solution $y(t)$ of the differential equation $y' = -y$ with initial condition $y(0) = c$ for three cases, $c = 1, 2, -1$. Sketch the solutions on the same graph for $t \geq 0$.

6. A bacteria culture starts with 500 cells and grows at a rate proportional to its size. After 3 hours there are 800 cells. a) Find an expression for the number of cells after t hours. b) Find the number of cells after 6 hours. c) When will the cell count reach 2048? Solve this problem two ways, with and without using a calculator (in the latter case, follow the alternative method shown in class).

7. Find the antiderivative.

a) $\int \sinh x dx$ b) $\int \cosh x dx$ c) $\int \tanh x dx$ d) $\int \text{sech } x dx$, where $\text{sech } x = \frac{1}{\cosh x}$

8. When a function value $f(x)$ is hard to compute, it can sometimes be approximated by another function value $f(a)$ which is easy to compute, where a is close to x , for example, $\sqrt{10} \approx \sqrt{9} = 3$. This is an example of Taylor approximation. The error is $|f(x) - f(a)|$ and it can be analyzed as follows. Assume $x \geq a$ for simplicity. Using the FTC in the form, $f(x) - f(a) = \int_a^x f'(t)dt$, it follows that

$$|f(x) - f(a)| = \left| \int_a^x f'(t)dt \right| \leq \int_a^x |f'(t)|dt \leq \int_a^x M_1 dt = M_1|x - a|, \quad (1)$$

where $M_1 = \max |f'(t)|$ for $a \leq t \leq x$. Equation (1) is an error bound; it implies that if x is close to a , then $f(x)$ is close to $f(a)$.

a) Let $f(x) = e^x, a = 0$. Sketch $y = f(x)$ and $y = f(a)$ on the same graph around $x = a$.

b) Make a table as follows, column 1: $|x - a|$, column 2: $|f(x) - f(a)|$, and fill it in for $x = 1, 1/2, 1/4, 1/8$ using a calculator; write the results with 4 decimal digits. When $|x - a|$ is reduced by a factor of $1/2$, by approximately what factor is the error $|f(x) - f(a)|$ reduced? Are the results consistent with the error bound in Equation (1)?