Math 156 Applied Honors Calculus II Fall 2024

hw7 , due: Tuesday, October 22 at 4pm , reminder : write solutions neatly and legibly

1. Consider the region between the curves $y = x^m$ and $y = x^n$, for $0 \le x \le 1$, where m, n are integers with $0 \le m < n$. (a) Sketch the region and label the curves for general m and n. (b) Find the center of mass of the region $(\bar{x}, \bar{y})$ in terms of m and n. (c) Consider the case $n = m+1$. Make a table as follows, column 1: m, column 2: n, column 3: $\bar{x}$, column 4: $\bar{x}^n$, column 5: $\bar{y}$, column 6: $\bar{x}^m$, column 7: CM lies in region? (yes or no). Take $m = 0, 1, 2, 3$. In columns 3-6, give the value in fractional form and decimal form with 4 decimal digits (e.g. $\frac{1}{3} = 0.3333$). Show that the CM lies inside the region for $m = 0, 1, 2$ and outside for $m = 3$.

2. Let X be a random variable with pdf $f(x) = \frac{3}{64} x\sqrt{16 - x^2}$ for $0 \le x \le 4$ and $f(x) = 0$ for all other x. (a) Sketch the graph of $f(x)$. (b) Verify that $f(x)$ is a valid pdf. (c) Find prob$(0 \le X \le 2)$.

3. The pdf of a normal distribution is $f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2}$ with constants $\mu, \sigma > 0$.
Show that (a) $\displaystyle\int_{-\infty}^{\infty} f(x)dx = 1$, (b) $\displaystyle\int_{-\infty}^{\infty} x f(x)dx = \mu$, (c) $\displaystyle\int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \sigma^2$.
This verifies that μ is the mean and σ is the standard deviation.
(hint: substitute $t = (x-\mu)/\sqrt{2\sigma^2}$, use the fact that $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$)

4. In class we considered the waiting time in the supermarket checkout line as a random variable T with exponential pdf $f(t)$. Assume the average waiting time is $\mu = 5$ minutes and show that the median waiting time is $m = 3.5$ minutes. Explain how it is possible for the average waiting time to be longer than the median waiting time.

5. Use separation of variables to find the solution $y(t)$ of the differential equation $y' = -y$ with initial condition $y(0) = c$ for three cases, $c = 1, 2, -1$. Sketch the solutions on the same graph for $t \ge 0$.

6. A bacteria culture starts with 500 cells and grows at a rate proportional to its size. After 3 hours there are 800 cells. a) Find an expression for the number of cells after t hours. b) Find the number of cells after 6 hours. c) When will the cell count reach 2048? Solve parts (a,b,c) two ways, with and without using a calculator (in the latter case, follow the alternative method shown in class).

7. Find the antiderivative.
a) $\int \sinh x\, dx$ b) $\int \cosh x\, dx$ c) $\int \tanh x\, dx$ d) $\int \operatorname{sech} x\, dx$, where $\operatorname{sech} x = \frac{1}{\cosh x}$

8. When a function value $f(x)$ is hard to compute, it can sometimes be approximated by another function value $f(a)$ which is easy to compute, where a is close to x, for example $\sqrt{10} \approx \sqrt{9} = 3$. This is an example of <u>Taylor approximation</u>. The <u>error</u> is $|f(x) - f(a)|$ and it can be analyzed as follows. Assume $x \ge a$ for simplicity. Using the FTC in the form, $f(x) - f(a) = \int_a^x f'(t)dt$, it follows that

$$|f(x) - f(a)| = \left|\int_a^x f'(t)dt\right| \le \int_a^x |f'(t)|dt \le \int_a^x M_1 dt = M_1|x-a|, \tag{1}$$

where $M_1 = \max|f'(t)|$ for $a \le t \le x$. Equation (1) is an <u>error bound</u>; it implies that if x is close to a, then $f(x)$ is close to $f(a)$.

a) Let $f(x) = e^x, a = 0$. Sketch $y = f(x)$ and $y = f(a)$ on the same graph around $x = a$.

b) Make a table as follows, column 1: $|x-a|$, column 2: $|f(x) - f(a)|$, and fill it in for $x = 1, 1/2, 1/4, 1/8$ using a calculator; write the results with 4 decimal digits. When $|x-a|$ is reduced by a factor of 1/2, by approximately what factor is the error $|f(x) - f(a)|$ reduced? Are the results consistent with the error bound in Equation (1)?