Math 156 Applied Honors Calculus II Fall 2023

hw9 , due: Friday, November 10 at 4pm

1. The hydrogen atom has one proton and one electron. In quantum mechanics, the ground state electron radius is a random variable with pdf $p(r) = (4/a_0^3)r^2e^{-2r/a_0}$, where $r \geq 0$ is distance from the proton and $a_0 = 5.29 \times 10^{-11}$m is the Bohr radius. (a) Show that $\lim_{r\to\infty} p(r)=0$. (b) At what radius does $p(r)$ attain its maximum value? Explain why your answer is a maximum rather than a minimum. (c) Sketch $p(r)$ for $r \geq 0$. (d) Verify that $p(r)$ is a pdf. (hint: substitute $t = 2r/a_0$) (e) Find the mean radius of the electron; give the answer in units of a_0. (f) Find the probability that the electron lies in the sphere of radius $4a_0$ centered on the proton.

2. A lake is initially stocked with 400 fish and the population triples after one year. Based on the lake size and nutrients available, the maximum capacity of the lake is estimated to be 10,000 fish. a) Using the logistic equation, find an expression for the fish population after t years. b) How long will it take for the fish population to reach 5,000?

3. Consider the differential equation $y' = -y$ with initial condition $y_0 = 1$. Solve the equation numerically for $y(1)$ by Euler's method with $t_n = n\Delta t = 1$. Make a table with the following entries. column 1: Δt (time step, take $\Delta t = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$); column 2: u_n (numerical solution at time t_n); column 3: $|y_n - u_n|$ (error); column 4: $|y_n - u_n|/\Delta t$ (ratio of error to time step) a) Indicate the limit of each column as in class (page 48 of notes). b) If the time step Δt is reduced by a factor of $\frac{1}{2}$, by approximately what factor is the error reduced?

4. Determine whether the series converges or diverges; justify your answer; if the series converges, find the sum. a) $3 + 2 + \frac{4}{3} + \frac{8}{9} + \cdots$ b) $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - 1 + \cdots$ c) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$ d) $\sum_{n=1}^{\infty} \frac{1}{3n+1}$

5. a) On hw8 you showed that $f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt$.

Now show that $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \int_a^x \frac{(x-t)^2}{2} f'''(t)\,dt$. (hint: in the integral from the hw8 result, integrate by parts with $u = f''(t)$, $dv = (x-t)\,dt$, and take $v = \frac{(x-t)^2}{-2}$)

b) Given a function $f(x)$ and a point a, define $T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$; then $T_2(x)$ is a quadratic function of x called the Taylor polynomial of degree 2. Show that $T_2(a) = f(a)$, then compute $T_2'(x), T_2''(x)$ and show that $T_2'(a) = f'(a), T_2''(a) = f''(a)$; hence $T_2(x)$ is a quadratic approximation to $f(x)$ at $x = a$.

c) Using (a,b) derive the error bound, $|f(x) - T_2(x)| \leq \frac{1}{3!}M_3|x-a|^3$, where $M_3 = \max|f'''(t)|$. (hint: (a,b) $\Rightarrow f(x) = T_2(x) + \int_a^x \frac{(x-t)^2}{2} f'''(t)\,dt$, then follow the steps from hw7, problem 8)

d) Let $f(x) = e^x$, $a = 0$. Find $T_2(x)$. Sketch $f(x), T_2(x)$ on the same graph around $x = a$; label both curves.

e) Make a table with the following format, column 1: $|x-a|$, column 2: $|f(x) - T_2(x)|$, and fill in the entries for $x = 1, 1/2, 1/4, 1/8$ using a calculator; write the results with 4 decimal digits. When $|x-a|$ is reduced by a factor of $\frac{1}{2}$, by approximately what factor is the error $|f(x) - T_2(x)|$ reduced? Is this consistent with the error bound derived in part (c)?

announcement. The 2nd midterm exam is on Wed Nov 8 at 6-7:30pm in 140 Lorch Hall. If you need accommodation, please tell your instructor. The exam covers 1.7 arclength, 1.8 surface area, 1.9 center of mass, 1.10 probability, 2.1 differential equations, 2.2 exponential growth/decay, 2.3 logistic equation, 2.4 Euler's method, 3.1 sequences, 3.2 series, 3.3 convergence tests for series, hyperbolic functions, Taylor polynomials, plus the homework and lecture notes. Calculators are not allowed, but you may use one sheet of paper (one side) for handwritten notes. At the end of the exam, scan and upload your solutions into Gradescope.