Math 156 Applied Honors Calculus II Fall 2024

hw10 , due: Monday, November 25 at 4pm

1. The age of organic remains can be determined by radiocarbon dating. When cosmic rays enter the atmosphere, they convert nitrogen to ^{14}C, a radioactive isotope of carbon with half-life 5730 years. Living animals absorb ^{14}C from the food chain and when they die the ^{14}C in their remains decreases by radioactive decay. Parchment is a thin sheet of organic material made from animal skins used for writing in ancient times. If a parchment fragment is discovered having 74% as much ^{14}C as in current living animal skins, what is the age of the fragment?

2. Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots$ by the following two methods.

method a) Express $\displaystyle\frac{1}{n(n+1)}$ by partial fractions, find a formula for the partial sum s_n, ...

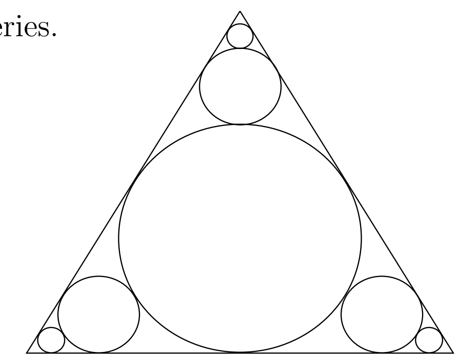
method b) Let a_n be the area of the region $\{(x,y) : 0 \le x \le 1, x^n \le y \le x^{n-1}\}$ for $n \ge 1$; sketch the regions for $n = 1,2,3,4$ on a common plot; compute a_n in terms of n for general n; relate the series to the area of the unit square.

3. a) $\sum_{n=1}^{\infty} \frac{1}{n^4}$, b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ Show that the series converges; how large must n be to ensure the error in the nth partial sum, $|s - s_n|$, is less than 10^{-6}? Use the error bounds derived in class.

4. Find the interval of convergence and the sum of the power series.

a) $\displaystyle\sum_{n=0}^{\infty} (x-4)^n$ b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{n x^n}{3^n}$

5. Three infinite sequences of tangent circles approach the vertices of an equilateral triangle (the figure shows the first few circles). Assume the triangle has sides of length 1. Express the area covered by the circles as an infinite series and find the sum. What fraction of the triangle area is covered by the circles?



6. In class we showed that $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$ for $-1 < x < 1$ by squaring the geometric series. Rederive this using the method of undetermined coefficients as follows;

$$\frac{1}{(1-x)^2} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \Rightarrow 1 = (1-x)^2(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$$
$$= (1 - 2x + x^2)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$$

Then multiply out the right side, collect powers of x, and solve for c_0, then c_1, then c_2, then c_3.

7. a) On hw9 you showed that $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \int_a^x \frac{(x-t)^2}{2} f'''(t)\,dt$.

Now show that $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \int_a^x \frac{(x-t)^3}{3!} f^{(4)}(t)\,dt$, where $f^{(4)}(t)$ is the 4th derivative. (hint: in the integral from the hw9 result, integrate by parts with $u = f'''(t)$, $dv = \frac{(x-t)^2}{2}dt$, and take $v = -\frac{(x-t)^3}{3!}$)

b) Given $f(x)$ and a point a, define $T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$; then $T_3(x)$ is a cubic function of x called the Taylor polynomial of degree 3. Show that $T_3(a) = f(a)$, then compute $T_3'(x), T_3''(x), T_3'''(x)$, and show that $T_3'(a) = f'(a)$, $T_3''(a) = f''(a)$, $T_3'''(a) = f'''(a)$; hence $T_3(x)$ is a cubic approximation to $f(x)$.

c) Using (a,b), derive the error bound $|f(x) - T_3(x)| \le \frac{1}{4!} M_4 |x-a|^4$, where $M_4 = \max |f^{(4)}(t)|$. (hint: (a,b) $\Rightarrow f(x) = T_3(x) + \int_a^x \frac{(x-t)^3}{3!} f^{(4)}(t)\,dt$; then follow the steps from hw7, problem 8)

d) Let $f(x) = e^x$, $a = 0$. Find $T_3(x)$. Sketch $f(x), T_3(x)$ on the same graph around $x = a$.

e) Make a table with the following format. column 1: $|x-a|$, column 2: $|f(x) - T_3(x)|$. Take $f(x) = e^x, a = 0$ and fill in the entries for $x = 1, 1/2, 1/4, 1/8$ using a calculator; write the results with 4 decimal digits. When $|x-a|$ is reduced by a factor of $\frac{1}{2}$, by approximately what factor is the error $|f(x) - T_3(x)|$ reduced? Is this consistent with the error bound derived in part (c)?