preview : integrals, differential equations, series

1
Mon
8/26

1 integrals

goal : prepare for FTC

1.1 sigma notation

definition : $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$: sum , series

i : index , a_i : terms , $1, n$: limits

example

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{i=1}^{5} i^2 = 1 + 4 + 9 + 16 + 25 = 55$$

note : A series can be written in different ways.

example

$$\sum_{i=1}^{5} i = \sum_{j=0}^{4} (j+1) = 1 + 2 + 3 + 4 + 5 = 15$$

set $j = i - 1$

properties of series

1. $\sum_{i=1}^{n} ca_i = c \cdot \sum_{i=1}^{n} a_i$, where c is any constant

2. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

3. $\sum_{i=1}^{n} a_i b_i \neq \sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i$ (but could be equal in some special cases)

proof

1. $\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = c \cdot \sum_{i=1}^{n} a_i$ ok

2. hw

3. consider $n = 2$

$$\sum_{i=1}^{2} a_i b_i = a_1 b_1 + a_2 b_2$$

$$\sum_{i=1}^{2} a_i \cdot \sum_{i=1}^{2} b_i = (a_1 + a_2) \cdot (b_1 + b_2) = a_1 b_1 + \underline{a_1 b_2 + a_2 b_1} + a_2 b_2$$ ok

<u>theorem</u>

1. $\sum_{i=1}^{n} 1 \;=\; 1 \,+\, 1 \,+\, 1 \,+\, \cdots \,+\, 1 \;=\; n$

2. $\sum_{i=1}^{n} i \;=\; 1 \,+\, 2 \,+\, 3 \,+\, \cdots \,+\, n \;=\; \dfrac{n(n+1)}{2}$, e.g. $n = 5 \overset{\substack{\text{implies}\\ \downarrow}}{\Rightarrow} S = 15$

3. $\sum_{i=1}^{n} i^2 \;=\; 1 \,+\, 4 \,+\, 9 \,+\, \cdots \,+\, n^2 \;=\; \dfrac{n(n+1)(2n+1)}{6}$, e.g. $n = 5 \Rightarrow S = 55$

<u>proof</u>

1. <u>ok</u>

2. $(i+1)^2 - i^2 \;=\; \cancel{i^2} + 2i \,+\, 1 \,-\, \cancel{i^2} \;=\; 2i \,+\, 1$

$$\sum_{i=1}^{n}((i+1)^2 - i^2) \;=\; \sum_{i=1}^{n}(2i+1) \;=\; 2\sum_{i=1}^{n} i \;+\; \sum_{i=1}^{n} 1 \;=\; \boxed{2S \,+\, n} \;, \text{ where } S = \sum_{i=1}^{n} i$$

$$\begin{aligned}\sum_{i=1}^{n}((i+1)^2 - i^2) \;&=\; (\cancel{2^2} - 1^2) \,+\, (\cancel{3^2} - \cancel{2^2}) \,+\, (\cancel{4^2} - \cancel{3^2}) \,+\, \cdots \,+\, ((n+1)^2 - \cancel{n^2}) \\ &=\; (n+1)^2 \,-\, 1^2 \;:\; \text{example of a } \underline{\text{telescoping series}} \\ &=\; \boxed{n^2 \,+\, 2n}\end{aligned}$$

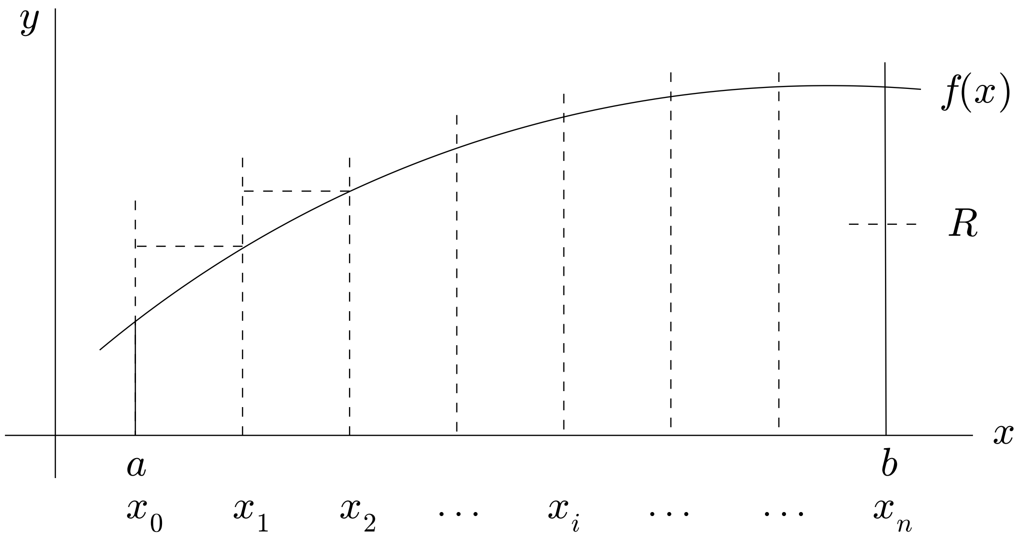
$\Rightarrow 2S + n \;=\; n^2 + 2n \;\Rightarrow\; 2S \;=\; n^2 + n \;=\; n(n+1) \;\Rightarrow\; S = \dfrac{n(n+1)}{2}$ <u>ok</u>

3. $(i+1)^3 - i^3 \;=\; \cdots$, hw

1.2 area

2 Tues 8/27

Given $f(x) \geq 0$, $a \leq x \leq b$.



R = region in the xy-plane between $y = 0$ and $y = f(x)$ for $a \leq x \leq b$

$= \{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$

problem : find the area of R

idea : approximate by rectangles

choose $n \geq 1$, set $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$, $i = 0, \ldots, n$

$x_0 = a$

$x_1 = a + \Delta x$

$x_2 = a + 2\Delta x$

$\ldots$

$x_n = a + n\Delta x = a + b - a = b$

area of ith rectangle $= f(x_i)\Delta x$ for $i = 1, \ldots, n$

area of region $R \approx \sum_{i=1}^{n} f(x_i)\Delta x$

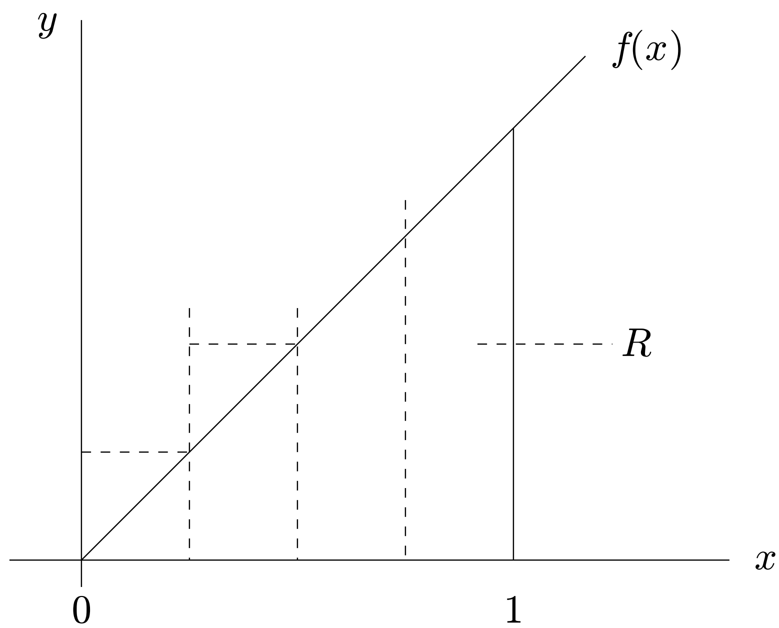
$\uparrow$

approximately

area of region $R = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

<u>example</u>

$f(x) = x \ , \ 0 \le x \le 1$



$R = \{(x, y) : 0 \le x \le 1, 0 \le y \le x\}$

of course , area $= \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2}$

$a = 0 \ , \ b = 1 \ , \ \Delta x = \frac{b-a}{n} = \frac{1}{n}$

$x_i = a + i\Delta x = \frac{i}{n}$

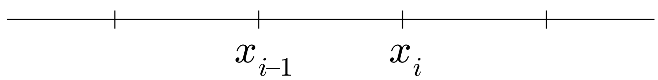
$f(x_i) = x_i = \frac{i}{n}$

$$\text{area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{n+1}{2n} = \frac{\infty}{\infty} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2}$$ <u>ok</u>

1.3 definite integral

As before, given $f(x)$, $a \le x \le b$, set $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$, $i = 0, \ldots, n$.



Let x_i^* be any point such that $x_{i-1} \le x_i^* \le x_i$.

Then $\sum\limits_{i=1}^{n} f(x_i^*)\Delta x$ is a <u>Riemann sum</u>. (Bernhard Riemann, 1826-1866)

$x_i^* = x_i$: right-hand RS

$x_i^* = x_{i-1}$: left-hand RS

$x_i^* = \dfrac{x_{i-1}+x_i}{2}$: midpoint RS

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx \ : \ \underline{\text{definite integral}} \ = \begin{cases} \text{area , volume , } \ldots \\ \text{distance , work , } \ldots \\ \text{probability , } \ldots \end{cases}$$

$$\int_0^1 dx = \lim_{n\to\infty} \sum_{i=1}^{n} 1 \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{i=1}^{n} 1 = \lim_{n\to\infty} \frac{1}{n} \cdot n = \lim_{n\to\infty} 1 = 1$$

$$a = 0 \ , \ b = 1 \ , \ \Delta x = \frac{b-a}{n} = \frac{1}{n} \ , \ x_i = a + i\Delta x = \frac{i}{n} \ , \ f(x) = 1$$

$$\int_0^1 x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \cdots = \frac{1}{2}$$

$$f(x) = x \ , \ f(x_i) = x_i = \frac{i}{n}$$

$$\int_0^1 x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$ 3 Wed 8/28

$$f(x) = x^2 \ , \ f(x_i) = x_i^2 = \left(\frac{i}{n}\right)^2 \qquad = \lim_{n\to\infty} \frac{2n^3 + \cdots}{6n^3} = \frac{1}{3}$$

$$\int_0^1 e^x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} e^{i/n} \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(e^{1/n}\right)^i = \cdots = e - 1$$

$$f(x) = e^x \ , \ f(x_i) = e^{x_i} = e^{i/n}$$

$\uparrow$

geometric series (hw2)

<u>definition</u> : $R_n = \sum_{i=1}^{n} f(x_i)\Delta x$: right-hand RS , $\lim_{n\to\infty} R_n = \int_a^b f(x)\,dx$

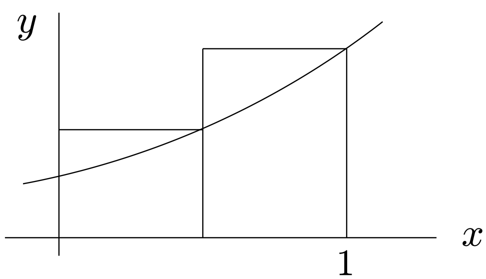
$M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$: midpoint RS , $\lim_{n\to\infty} M_n = \int_a^b f(x)\,dx$

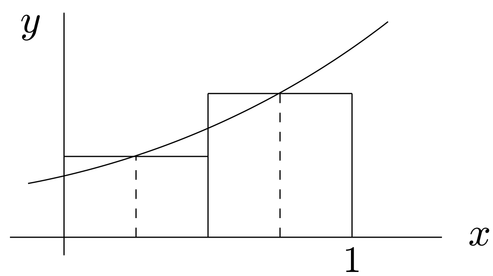
question : For a given value of n, which approximation is more accurate?

<u>example</u> : $\int_0^1 e^x dx = e - 1 = 1.71828183 = I$

n	Δx	R_n	$\lvert I - R_n\rvert$	M_n	$\lvert I - M_n\rvert$
1	1	2.7183	1.0000	1.6487	0.0696
2	0.5	2.1835	0.4652	1.7005	0.0178
4	0.25	1.9420	0.2237	1.7138	0.0045

Hence the midpoint RS is more accurate than the right-hand RS. Why? Consider $n = 2$.





If Δx decreases by a factor of $1/2$,
then the error in R_n decreases by a factor of approximately $1/2$,
and $\cdots\cdots$ " $\cdots\cdots$ M_n $\cdots\cdots\cdots\cdots\cdots\cdots$ " $\cdots\cdots\cdots\cdots\cdots\cdots$ $1/4 = (1/2)^2$.

<u>properties of the definite integral</u>

1. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
2. $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
4. If $f(x) \le g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

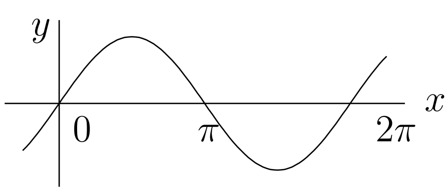
<u>proof</u>

1. $\int_a^b cf(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} cf(x_i)\Delta x = c\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = c\int_a^b f(x)\,dx$ <u>ok</u> 4 Fri 8/30

2. hw2 , 3. and 4. omit

<u>note</u> : If $f(x)$ changes sign, then $\int_a^b f(x)\,dx$ = <u>signed area</u>.

<u>example</u> : $f(x) = \sin x$

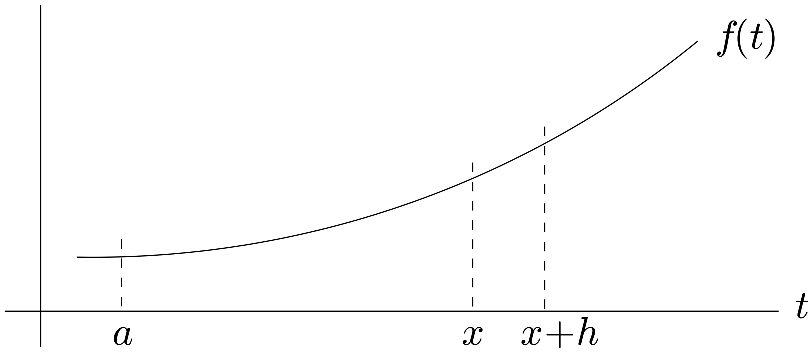


$\int_0^{\pi} \sin x\,dx > 0$, $\int_{\pi}^{2\pi} \sin x\,dx < 0$, $\int_0^{2\pi} \sin x\,dx = 0$

1.4 FTC

FTC, part 1 : $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$

proof



define : $F(x) = \displaystyle\int_a^x f(t)\,dt$, we need to show that $F'(x) = f(x)$

recall : $F'(x) = \displaystyle\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$

$$F(x+h) = \int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt = F(x) + \int_x^{x+h} f(t)\,dt$$

$\Rightarrow F(x+h) - F(x) = \displaystyle\int_x^{x+h} f(t)\,dt \approx f(t^*)\cdot h$, where $x \le t^* \le x+h$

$$\Rightarrow \frac{F(x+h)-F(x)}{h} \approx f(t^*)$$

$\Rightarrow F'(x) = \displaystyle\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(t^*) = f(x)$ ok (if f is continuous, Math 451)

example : $\dfrac{d}{dx}\displaystyle\int_0^x t\,dt = x$, check . . .

$$\int_0^x t\,dt = \lim_{n\to\infty}\sum_{i=1}^n t_i\Delta t = \lim_{n\to\infty}\sum_{i=1}^n \frac{ix}{n}\cdot\frac{x}{n} = x^2\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^n i = x^2\lim_{n\to\infty}\frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \frac{x^2}{2}$$

$a = 0$, $b = x$, $\Delta t = \dfrac{b-a}{n} = \dfrac{x}{n}$, $t_i = a + i\Delta t = \dfrac{ix}{n}$

$\Rightarrow \dfrac{d}{dx}\displaystyle\int_0^x t\,dt = \frac{d}{dx}\left(\frac{x^2}{2}\right) = x$ ok

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example (time permitting at end of Friday class)

$\{(x,y) : 1 \le x \le 2, 0 \le y \le x\}$, sketch region , find area by RS

5 Tues 9/3

FTC, part 2 : $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x)$

proof : $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$

$$\sum_{i=1}^n f(x_i)\Delta x = \sum_{i=1}^n F'(x_i)\Delta x \approx \sum_{i=1}^n \left(\frac{F(x_i+\Delta x) - F(x_i)}{\cancel{\Delta x}}\right)\cancel{\Delta x}$$

$$= \sum_{i=1}^n (F(x_i+\Delta x) - F(x_i)) \ , \ x_i + \Delta x = a + i\Delta x + \Delta x = a + (i+1)\Delta x = x_{i+1}$$

$$= \sum_{i=1}^n (F(x_{i+1}) - F(x_i))$$

$$= (F(\cancel{x_2}) - F(x_1)) + (F(\cancel{x_3}) - F(\cancel{x_2})) + \cdots + (F(x_{n+1}) - F(\cancel{x_n})) \ : \ \text{telescoping sum}$$

$$= F(x_{n+1}) - F(x_1) = F(b+\Delta x) - F(a+\Delta x)$$

$$x_1 = a + \Delta x$$

$$x_{n+1} = a + (n+1)\Delta x = a + n\Delta x + \Delta x = a + (b-a) + \Delta x = b + \Delta x$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty} (F(b+\Delta x) - F(a+\Delta x)) = F(b) - F(a)$$

ok (if f is continuous)

We write $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$.

If $F'(x) = f(x)$, we say that $F(x)$ is an antiderivative of $f(x)$.

Then $F(x)+c$ is also an antiderivative of $f(x)$ and we write $\int f(x)\,dx = F(x)+c$.

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recall : $\displaystyle\int_0^1 x\,dx = \lim_{n\to\infty} \sum_{i=1}^n x_i\Delta x = \lim_{n\to\infty} \sum_{i=1}^n \frac{i}{n}\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2}$

$a = 0$, $b = 1$, $\Delta x = \dfrac{b-a}{n} = \dfrac{1}{n}$, $x_i = a + i\Delta x = \dfrac{i}{n}$

now we can use the FTC : $\displaystyle\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

$$\int_0^1 \sqrt{x}\,dx = \lim_{n\to\infty} \sum_{i=1}^n \sqrt{x_i}\Delta x = \lim_{n\to\infty} \sum_{i=1}^n \left(\frac{i}{n}\right)^{1/2} \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^{3/2}} \sum_{i=1}^n i^{1/2} = \ ?$$

$$= \frac{2}{3} x^{3/2}\Big|_0^1 = \frac{2}{3}$$

$f(x)$	$F(x)$	
x^n	$\dfrac{x^{n+1}}{n+1}$, $\quad n \neq -1$	
x^{-1}	$\ln x$	
$\ln x$	$x \ln x - x$	
e^x	e^x	
$\sin x$	$-\cos x$	
$\cos x$	$\sin x$	
$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\cosh x$	more later
$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$\sinh x$	more later
$\dfrac{1}{x^2+1}$	$\tan^{-1} x$	
e^{x^2}	$\int_a^x e^{t^2} dt$: cannot be expressed in terms of elementary functions	

6
Wed
9/4

1.5 work

work = force × displacement (distance) , force = mass × acceleration

units	metric (SI)	British
mass	kilogram : kg	slug : ?
distance	meter : m	foot : ft
time	second : s	second : s
force	Newton : $\mathrm{N} = \mathrm{kg} \cdot \mathrm{m/s}^2$	pound : lb
work	Joule : $\mathrm{J} = \mathrm{N} \cdot \mathrm{m}$	foot-pound : ft-lb

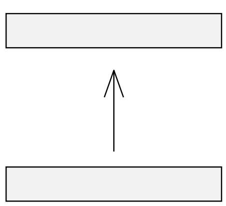
<u>conversion</u>

$1\,\mathrm{m} = 3.28\,\mathrm{ft}$, $1\,\mathrm{N} = 0.225\,\mathrm{lb}$ $\Rightarrow$ $1\,\mathrm{J} = 0.738\,\text{ft-lb}$

$g = 9.8\ \mathrm{m/s}^2 = 32.2\ \mathrm{ft/s}^2$

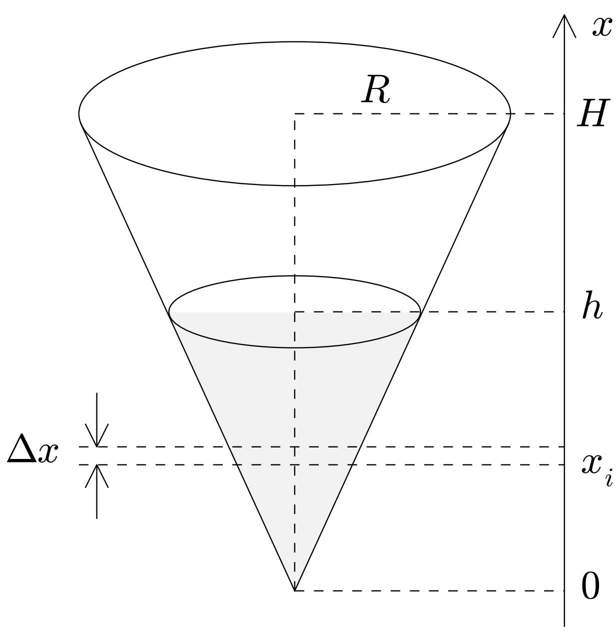
<u>example</u>

Find the work done in lifting a 1 kg book to a height of 1 m above a table.



$$W = \text{force} \times \text{distance} = mg \times d = 1\,\mathrm{kg} \times 9.8\,\mathrm{m/s}^2 \times 1\,\mathrm{m} = 9.8\,\mathrm{J}$$

<u>example</u> : A water tank has the shape of an inverted cone.



R : base radius of cone

H : height of cone

h : water level

ρ : water density

problem : Find the work done in pumping the water to the top of the tank.

idea : Think of the water volume as a stack of books.

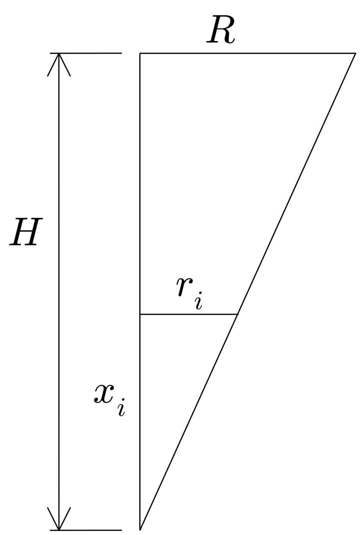
$\Delta x = \dfrac{h}{n}$: width of a water layer

$x_i = i\Delta x$: height of ith layer $(x_0 = 0\,,\, x_n = h)$

work = force $\times$ distance = mass $\times$ acceleration $\times$ distance

mass of water in ith layer = density $\times$ volume $\approx \rho \cdot \pi r_i^2 \Delta x$

r_i : radius of ith layer



$$\frac{R}{H} = \frac{r_i}{x_i} \;\Rightarrow\; r_i = \frac{R}{H}\,x_i$$

check : $x_i = 0 \;\Rightarrow\; r_i = 0$

$x_i = H \;\Rightarrow\; r_i = R$ <u>ok</u>

force acting on ith layer $\approx \rho\pi r_i^2 \Delta x \cdot g$

work done in raising ith layer $\approx \rho g \pi r_i^2 \Delta x \cdot (H - x_i)$

work done in raising entire water volume

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g \pi r_i^2 (H - x_i)\Delta x = \int_0^h \rho g \pi x^2 \frac{R^2}{H^2}(H-x)dx$$

$$\int_0^h x^2(H-x)dx = \int_0^h (x^2H - x^3)dx = \left(\frac{x^3}{3}H - \frac{x^4}{4}\right)\Bigg|_0^h = \frac{h^3}{3}H - \frac{h^4}{4} = \frac{h^3}{12}(4H - 3h)$$

$W = \rho g \pi \dfrac{R^2}{H^2} \cdot \dfrac{h^3}{12}(4H - 3h)$, check : $h = 0 \Rightarrow W = 0$ <u>ok</u>

plug in numbers

7 Fri 9/6

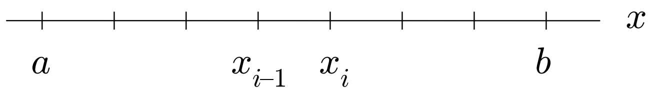
$R = 4\,\mathrm{m}$, $H = 10\,\mathrm{m}$, $h = 8\,\mathrm{m}$, $\rho = 1000\,\mathrm{kg/m^3}$, $g = 9.8\,\mathrm{m/s^2}$

$$W = 10^3\,(9.8)\,(3.14)\,\frac{16}{10^2} \cdot \frac{8^3}{12}\,(40 - 24)\,\frac{\mathrm{kg}}{\cancel{\mathrm{m}^3}}\,\frac{\mathrm{m}}{\mathrm{s}^2}\,\frac{\cancel{\mathrm{m}^2}}{\cancel{\mathrm{m}^2}}\,\cancel{\mathrm{m}^3} \cdot \mathrm{m}$$

$$\approx 10 \cdot 10 \cdot \cancel{3} \cdot \frac{4}{\cancel{3}} \cdot 5 \cdot 10^2 \cdot 16\,\mathrm{J} \;\approx\; 3.2 \cdot 10^6\,\mathrm{J} \;=\; 3.2\,\mathrm{MJ}$$

<u>note</u> : In these examples the force (due to gravity) is assumed to be constant as the mass is raised (book, water), but in general the force may depend on the displacement of the mass.

<u>example</u>: Compute the work done in moving an object from $x = a$ to $x = b$ subject to a variable force $f(x)$.



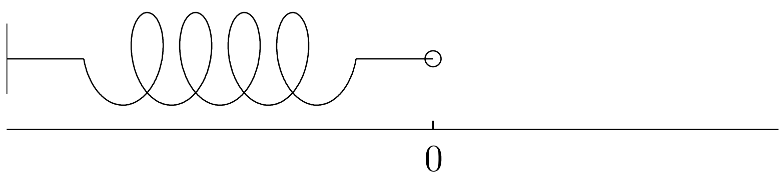
, $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$

work done in moving object from x_{i-1} to $x_i \approx f(x_i)\Delta x$

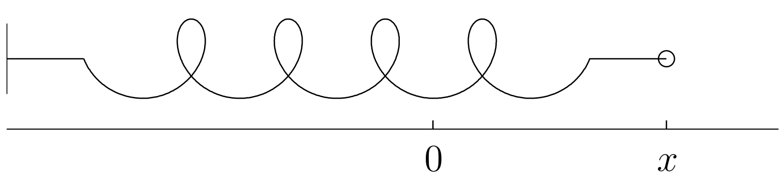
. . . . " . . . . a to $b \approx \sum_{i=1}^{n} f(x_i)\Delta x$

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

<u>example</u>: A spring is stretched x units from its natural length.



: natural length



: stretched length

<u>Hooke's law</u>

The force needed to stretch a spring is proportional to the displacement of the spring from its natural length.

$f(x) = kx$, k : <u>spring constant</u> , x : displacement from natural length

<u>example</u> : A 40 N force is needed to stretch a spring from its natural length of 10 cm to a length of 15 cm. Find the work done in stretching the spring from length 15 cm to 20 cm.

$$f(x) = kx \Rightarrow 40\,\mathrm{N} = k \cdot (15\,\mathrm{cm} - 10\,\mathrm{cm}) = k \cdot 5\,\mathrm{cm} \Rightarrow k = \frac{40\,\mathrm{N}}{5\,\mathrm{cm}} = 8\frac{\mathrm{N}}{\mathrm{cm}}$$

$$W = \int_a^b f(x)dx = \int_5^{10} kx dx = k\frac{x^2}{2}\Big|_5^{10} = 8\left(\frac{100}{2} - \frac{25}{2}\right) = 300\frac{\mathrm{N}}{\mathrm{cm}}\,\mathrm{cm}^2 = 3\,\mathrm{N\,m} = 3\,\mathrm{J}$$

<u>note</u> : Hooke's law is a <u>linear approximation</u> to the actual force.

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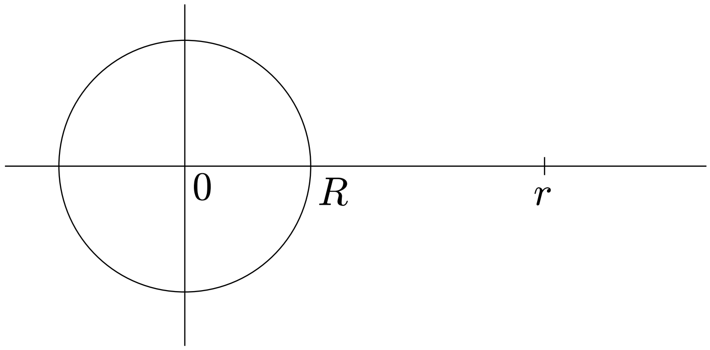
example

Find the work done in moving a particle from the Earth's surface to ∞.



R : radius of Earth

r : distance from particle to center of Earth

$f(r) = \dfrac{GMm}{r^2}$: force on particle due to gravity (Isaac Newton, 1643-1727)

G : gravitational constant

M : Earth mass

m : particle mass

$$W = \underbrace{\int_R^\infty f(r)dr}_{\uparrow} = \int_R^\infty GMm\frac{dr}{r^2} = GMm \cdot \frac{-1}{r}\Big|_R^\infty = GMm \cdot \left(0 - \frac{-1}{R}\right) = \frac{GMm}{R}$$

improper integral , more later

We can compute the escape velocity of the particle.

work = kinetic energy

$$\frac{GMm}{R} = \frac{1}{2}mv_{\text{esc}}^2 \;\Rightarrow\; v_{\text{esc}} = \left(\frac{2GM}{R}\right)^{1/2}$$

$$v_{\text{esc}} = \left(2 \cdot \frac{6.67 \cdot 10^{-11}\text{N} \cdot \text{m}^2/\text{kg}^2 \times 5.97 \cdot 10^{24}\text{kg}}{6.37 \cdot 10^6\,\text{m}}\right)^{1/2}$$

$$\approx 11\,\frac{\text{km}}{\text{s}} = 33 \cdot \text{speed of sound in air} = 3.67 \cdot 10^{-5} \cdot \text{speed of light in vacuum}$$

note : $R \to 0$: black hole , $v_{\text{esc}} \to \infty$: impossible (Albert Einstein, 1879-1955)

$\Rightarrow$ nothing can escape the gravitational field of a black hole

1.6 improper integrals

<u>definition</u>: $\int_a^b f(x)\,dx$ is an <u>improper integral</u> if $\begin{cases} a=-\infty \text{ or } b=\infty \\ \text{or} \\ f(x)\to\pm\infty \text{ in } (a,b) \end{cases}$

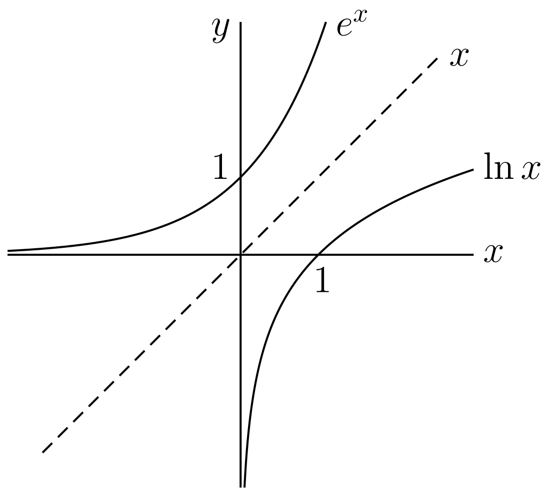
An improper integral is computed by taking a limit of proper integrals; if the limit is finite, the integral <u>converges</u>; otherwise, it <u>diverges</u>.

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = \lim_{b\to\infty}\Big(-\frac{1}{b}+1\Big) = 1 : \text{ converges}$$

$\Rightarrow$ the area under the graph of $y=\dfrac{1}{x^2}$ from $x=1$ to $x=\infty$ is finite

short cut : $\displaystyle\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^\infty = -\frac{1}{\infty}+1 = 1$

$$\int_1^\infty \frac{dx}{x} = \ln x\Big|_1^\infty = \ln\infty - \ln 1 = \infty \;:\; \text{diverges}$$



$\Rightarrow$ the area under the graph of $y=\dfrac{1}{x}$ from $x=1$ to $x=\infty$ is infinite

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$\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$: diverges

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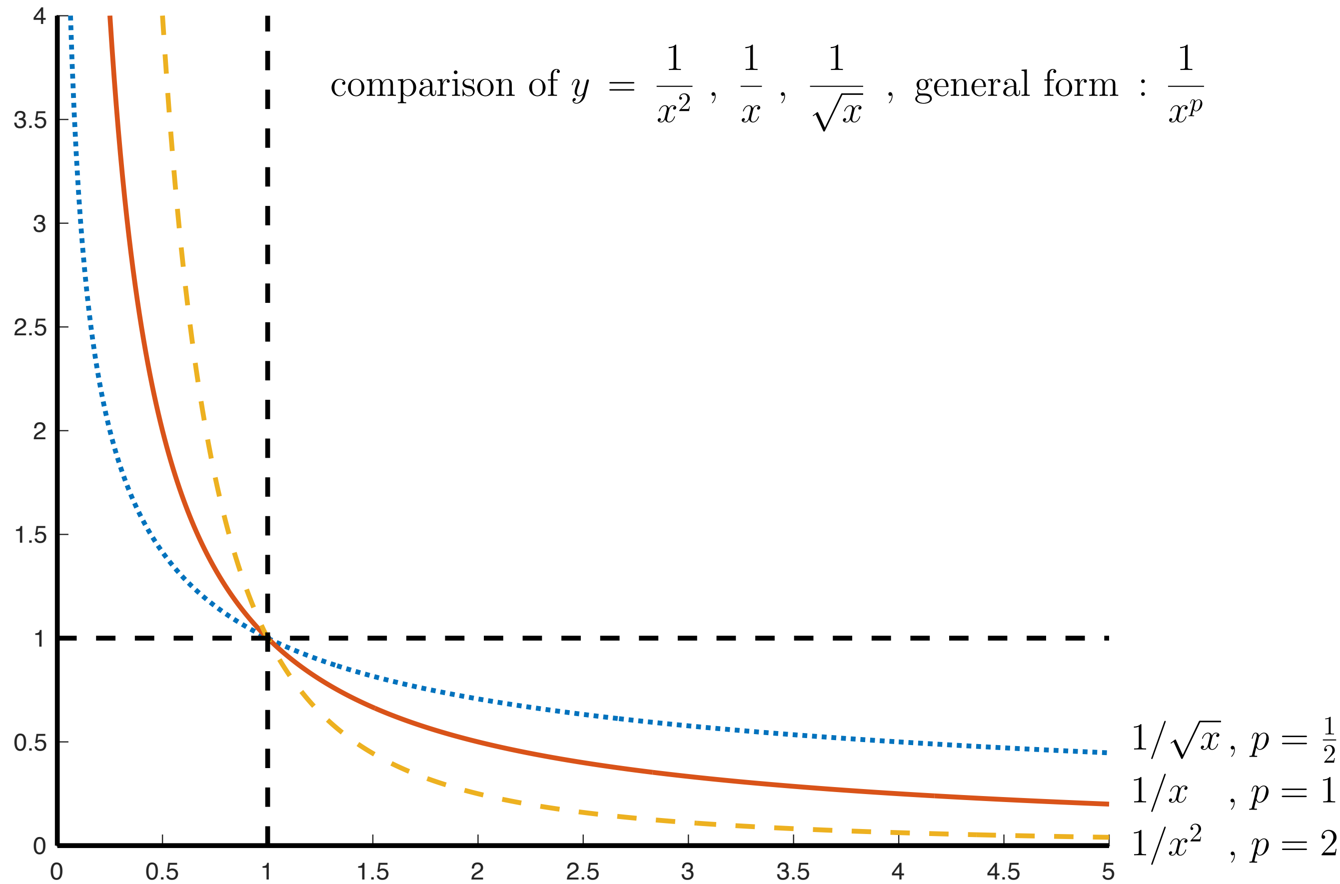
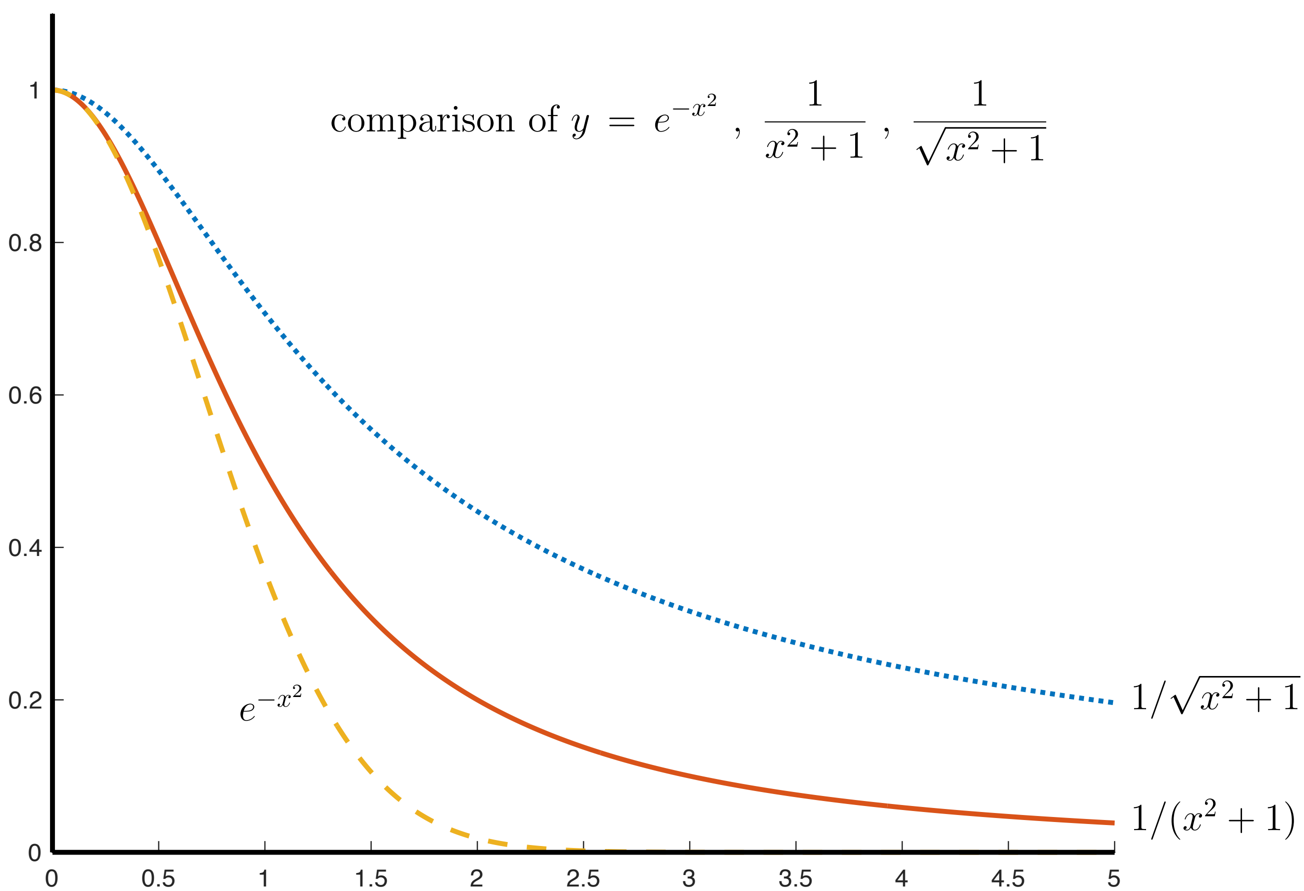
pf 1 : $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_1^\infty = 2\sqrt{\infty}-2 = \infty$: diverges

pf 2 : $x\ge 1 \Rightarrow x\ge\sqrt{x} \Rightarrow \dfrac{1}{x}\le\dfrac{1}{\sqrt{x}} \underset{\uparrow}{\Rightarrow} \displaystyle\int_1^\infty \frac{dx}{x} \le \int_1^\infty \frac{dx}{\sqrt{x}}$: diverges

<u>comparison test</u>



comparison of $y = \frac{1}{x^2}$, $\frac{1}{x}$, $\frac{1}{\sqrt{x}}$, general form : $\frac{1}{x^p}$
$1/\sqrt{x}$, $p = \frac{1}{2}$
$1/x$, $p = 1$
$1/x^2$, $p = 2$
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5



comparison of $y = e^{-x^2}$, $\frac{1}{x^2+1}$, $\frac{1}{\sqrt{x^2+1}}$
e^{-x^2}
$1/\sqrt{x^2+1}$
$1/(x^2+1)$
0
0.2
0.4
0.6
0.8
1
0.5
1.5
2
2.5
3
3.5
4
4.5
5

So far we considered $\int_1^\infty f(x)\,dx$, where $f(x) \to 0$ as $x \to \infty$.

Now consider $\int_0^1 f(x)\,dx$, where $f(x) \to \infty$ as $x \to 0$.

$$\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_0^1 = 2-0 = 2 \ : \ \text{converges}$$

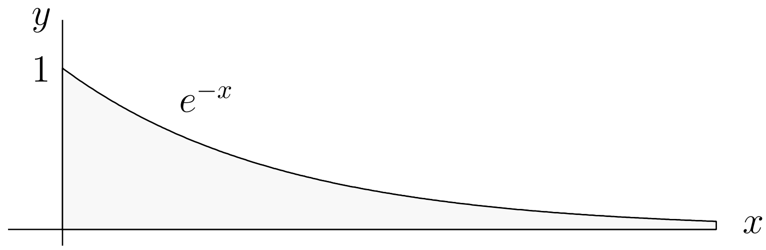
$$\int_0^1 \frac{dx}{x} = \ln x\Big|_0^1 = \ln 1 - \ln 0 = 0-(-\infty) = \infty \ : \ \text{diverges}$$

$$\int_0^1 \frac{dx}{x^2} = -\frac{1}{x}\Big|_0^1 = -\frac{1}{1} - \Big(-\frac{1}{0}\Big) = -1+\infty = \infty \ : \ \text{diverges (or comparison test)}$$

<u>p-test</u>

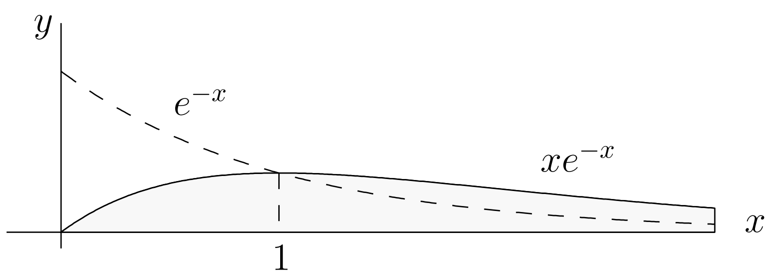
$$\int_1^\infty \frac{dx}{x^p} : \begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\le 1 \end{cases} , \quad \int_0^1 \frac{dx}{x^p} : \begin{cases} \text{converges if } p<1 \\ \text{diverges if } p\ge 1 \end{cases} , \quad \text{pf : omit}$$

$$\int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = -e^{-\infty} - (-e^0) = 0-(-1) = 1 \ : \ \text{converges}$$



$$\int_0^\infty x e^{-x} dx = 1 \ : \ \text{converges}$$

$$\lim_{x\to\infty} xe^{-x} = \infty\cdot 0 = \lim_{x\to\infty}\frac{x}{e^x} = \frac{\infty}{\infty} = \lim_{x\to\infty}\frac{1}{e^x} = 0 \ : \ \text{l'Hôpital's rule, proof later}$$



integration by parts : $\int_a^b u dv = uv\Big|_a^b - \int_a^b v du$

choose $u = x$, $dv = e^{-x}dx \ \Rightarrow \ du = dx$, $v = -e^{-x}$

$$\Rightarrow \int_0^\infty xe^{-x}dx = \cancel{-xe^{-x}}\Big|_0^\infty + \int_0^\infty e^{-x}dx = 0+1 = 1$$

What happens if we choose $u = e^{-x}$, $dv = xdx$? . . .

<u>integration by parts</u>

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$(u(x)v(x))' = u(x)v'(x) + u'(x)v(x)$

$(uv)' = uv' + u'v$

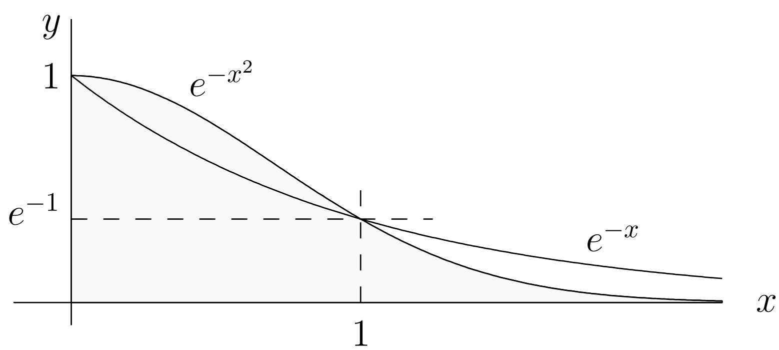
$\Rightarrow \int_a^b (uv)'dx = \int_a^b uv'dx + \int_a^b u'vdx$

$\Rightarrow uv\Big|_a^b = \int_a^b udv + \int_a^b vdu$

$v'dx = \frac{dv}{\cancel{dx}}\cancel{dx} = dv \ , \ u'dx = \frac{du}{\cancel{dx}}\cancel{dx} = du$

$\Rightarrow \int_a^b udv = uv\Big|_a^b - \int_a^b vdu$ <u>ok</u>

$\int_0^\infty e^{-x^2}dx$: converges



The antiderivative is not an elementary function, so we need a different approach.

$0 \le x \le 1 \Rightarrow e^{-x^2} \le 1$

$1 \le x < \infty \Rightarrow x^2 \ge x \Rightarrow -x^2 \le -x \Rightarrow e^{-x^2} \le e^{-x}$

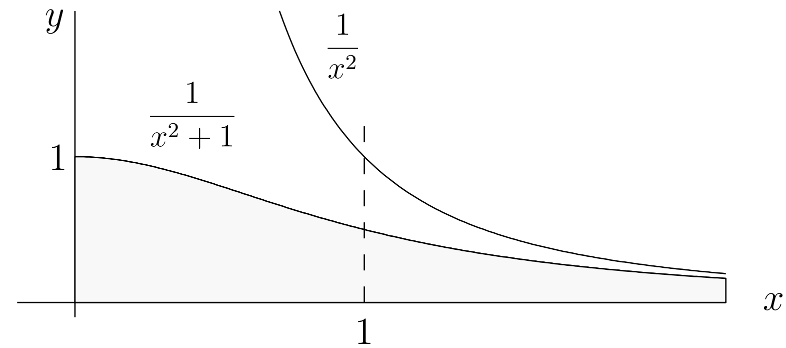
(this is because e^x is an <u>increasing</u> function, i.e. $a \le b \Rightarrow e^a \le e^b$)

$\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$

$\quad \le \int_0^1 1 \cdot dx + \int_1^\infty e^{-x}dx = 1 + (-e^{-x})\Big|_1^\infty = 1 + e^{-1} = 1.3679$ <u>ok</u>

note : $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} = 0.8862$, proof : Math 215 , multivariable calculus

$$\int_0^\infty \frac{dx}{x^2+1} \ : \ \text{converges}$$



$$\frac{1}{x^2+1} \le \frac{1}{x^2} \Rightarrow \int_0^\infty \frac{dx}{x^2+1} \le \int_0^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big|_0^\infty = -\frac{1}{\infty} + \frac{1}{0} = \infty$$

This yields no information, so we need a different approach.

$$\int_0^\infty \frac{dx}{x^2+1} = \int_0^1 \frac{dx}{x^2+1} + \int_1^\infty \frac{dx}{x^2+1} \le \int_0^1 1 \cdot dx + \int_1^\infty \frac{dx}{x^2} = 1 + 1 = 2 \quad \underline{\text{ok}}$$

<u>alternative</u>

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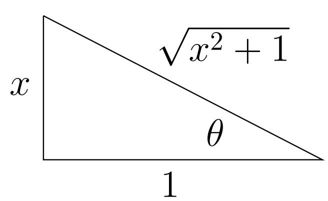
$$\int \frac{dx}{x^2+1} = \tan^{-1} x = \arctan x$$



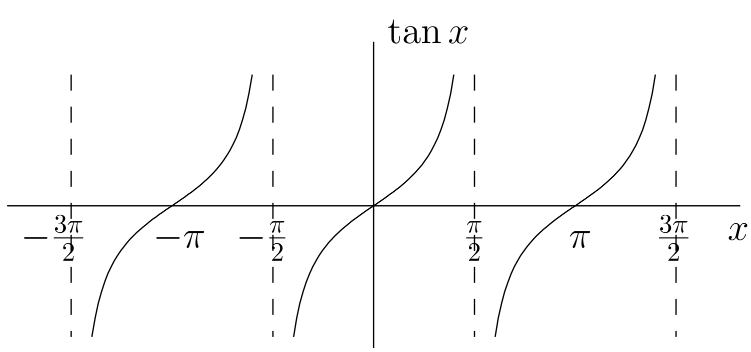
$\tan\theta = x$: trigonometric substitution

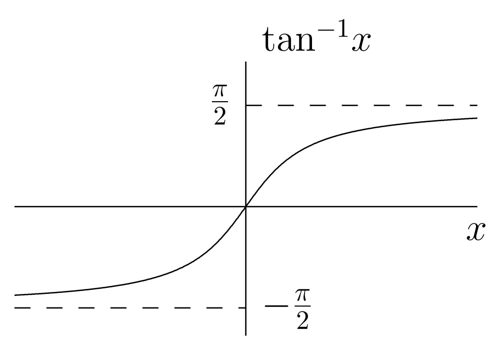
$\sec^2\theta \, d\theta = dx$

$\sec\theta = \sqrt{x^2+1}$

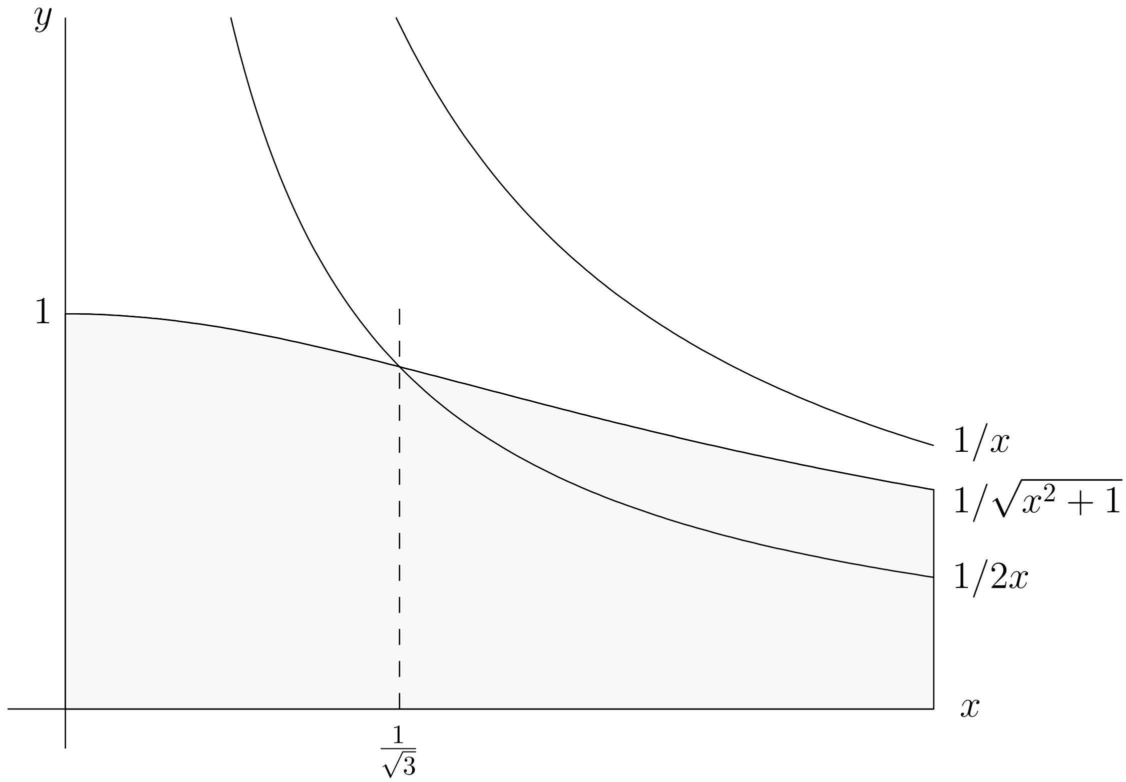
$$\int \frac{dx}{x^2+1} = \int \frac{\cancel{\sec^2\theta} \, d\theta}{\cancel{\sec^2\theta}} = \int d\theta = \theta = \tan^{-1} x$$

$$\int_0^\infty \frac{dx}{x^2+1} = \tan^{-1} x \Big|_0^\infty = \tan^{-1}\infty - \tan^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2} = 1.5708 \quad \underline{\text{ok}}$$





$\int_0^\infty \frac{dx}{\sqrt{x^2+1}}$: diverges



$$\frac{1}{\sqrt{x^2+1}} \leq \frac{1}{\sqrt{x^2}} = \frac{1}{x} \;\Rightarrow \int_0^\infty \frac{dx}{\sqrt{x^2+1}} \leq \int_0^\infty \frac{dx}{x} \;:\; \text{diverges}$$

This yields no information, so we need a different approach.

idea : $\underset{\substack{\uparrow \\ \underline{\text{asymptotic}}}}{\frac{1}{\sqrt{x^2+1}} \sim \frac{1}{x}}$ as $x \to \infty$, so we still expect that $\int_0^\infty \frac{dx}{\sqrt{x^2+1}}$ diverges

<u>definition</u>: $f(x) \sim g(x)$ as $x \to \infty$ means $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

To prove that the integral diverges, we use a reverse inequality.

$$\frac{1}{\sqrt{x^2+1}} \geq \frac{1}{2x} \underset{\substack{\uparrow \\ \underline{\text{if and only if}}}}{\Leftrightarrow} 2x \geq \sqrt{x^2+1} \Leftrightarrow 4x^2 \geq x^2+1 \Leftrightarrow 3x^2 \geq 1 \Rightarrow x \geq \frac{1}{\sqrt{3}}$$

$$\int_0^\infty \frac{dx}{\sqrt{x^2+1}} = \underset{\substack{\uparrow \\ \text{converges}}}{\int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{\sqrt{x^2+1}}} + \underset{\substack{\uparrow \\ \text{diverges}}}{\int_{\frac{1}{\sqrt{3}}}^\infty \frac{dx}{\sqrt{x^2+1}}}$$

, $\int_{\frac{1}{\sqrt{3}}}^\infty \frac{dx}{\sqrt{x^2+1}} \geq \int_{\frac{1}{\sqrt{3}}}^\infty \frac{dx}{2x}$: diverges <u>ok</u>

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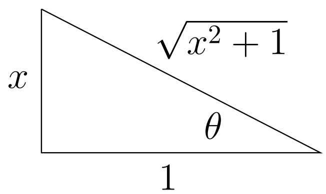
<u>alternative</u>



$$\tan\theta = x$$
$$\sec^2\theta\, d\theta = dx$$
$$\sec\theta = \sqrt{x^2+1}$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \int \frac{\sec^2\theta}{\sec\theta} d\theta = \int \sec\theta d\theta = \ ?$$

$$\int \sec\theta d\theta = \int \frac{1}{\cos\theta} d\theta = \int \frac{\cos\theta}{\cos^2\theta} d\theta = \int \frac{\cos\theta}{1-\sin^2\theta} d\theta = \int \frac{du}{1-u^2} = \ ?$$

$$u = \sin\theta \ \Rightarrow \ du = \cos\theta d\theta$$

<u>partial fractions</u>

idea : $\dfrac{3}{10} = \dfrac{3}{2\cdot 5} = \dfrac{5-2}{2\cdot 5} = \dfrac{1}{2} - \dfrac{1}{5}$

$$\frac{1}{1-u^2} = \frac{1}{(1+u)(1-u)} = \frac{a}{1+u} + \frac{b}{1-u} = \frac{a(1-u)+b(1+u)}{(1+u)(1-u)}$$

$$= \frac{(a+b)+u(-a+b)}{1-u^2}$$

$$\Rightarrow \ (a+b)+u(-a+b) = 1 \text{ for all } u \ \Rightarrow \ a+b=1\,,\ -a+b=0 \ \Rightarrow \ a=b=\frac{1}{2}$$

$$\Rightarrow \ \frac{1}{1-u^2} = \frac{1/2}{1+u} + \frac{1/2}{1-u} \quad , \ \text{check} \ \ldots$$

$$\int \frac{du}{1-u^2} = \frac{1}{2}\int \frac{du}{1+u} + \frac{1}{2}\int \frac{du}{1-u} = \frac{1}{2}\ln(1+u) - \frac{1}{2}\ln(1-u) = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$$

recall : $\ln a + \ln b = \ln(ab)$, $\ln a - \ln b = \ln(a/b)$, $a\ln b = \ln(b^a)$

$$\int \sec\theta d\theta = \frac{1}{2}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right) = \frac{1}{2}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}\right) = \frac{1}{2}\ln\left(\frac{(1+\sin\theta)^2}{1-\sin^2\theta}\right)$$

$$= \frac{1}{2}\ln\left(\frac{(1+\sin\theta)^2}{\cos^2\theta}\right) = \ln\left(\frac{1+\sin\theta}{\cos\theta}\right) = \ln(\sec\theta + \tan\theta)$$

$$\int \sec\theta d\theta = \ln(\sec\theta+\tan\theta) \quad , \ \text{check} \ \ldots$$

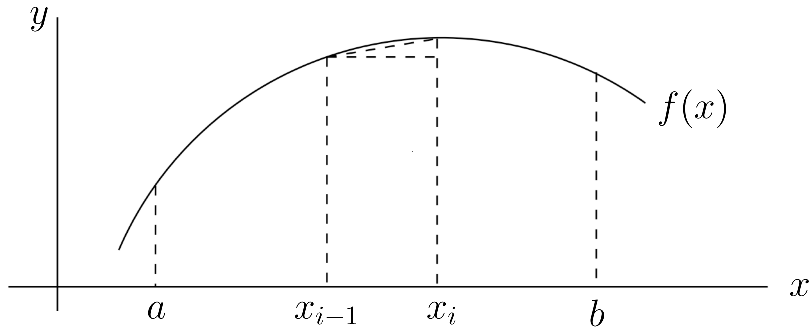
$$\int \frac{dx}{\sqrt{x^2+1}} = \ln(\sqrt{x^2+1}+x) \quad , \ \text{check} \ \ldots$$

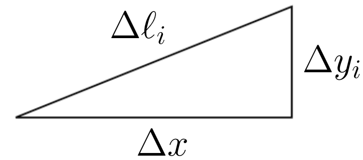
$$\int_0^\infty \frac{dx}{\sqrt{x^2+1}} = \ln(\sqrt{x^2+1}+x)\Big|_0^\infty = \ln\infty - \ln 1 = \infty \ : \ \text{diverges} \qquad \underline{\text{ok}}$$

another alternative : ChatGPT

1.7 arclength

problem : find the length of the graph of a function





example : trajectory of a rocket, charged particle in a magnetic field, ...

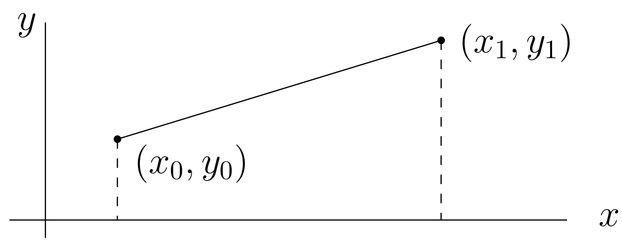
idea : divide the graph into short segments

$$\Delta x = \frac{b-a}{n}\ ,\ x_i = a + i\Delta x\ ,\ \Delta y_i = f(x_i) - f(x_{i-1})$$

$$\Delta\ell_i = \sqrt{\Delta x^2 + \Delta y_i^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2} \cdot \Delta x \approx \sqrt{1 + (f'(x_i))^2} \cdot \Delta x$$ 13 Tues 9/17

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta\ell_i = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i))^2}\, \Delta x = \int_a^b \sqrt{1 + (f'(x)^2)}\, dx$$

<u>example</u> : straight line



recall : $L = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

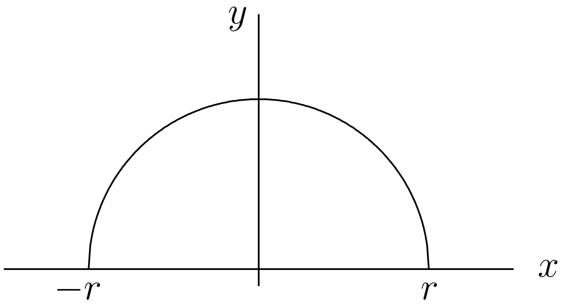
$$y = f(x) = mx + b \ \Rightarrow\ f'(x) = m = \frac{y_1 - y_0}{x_1 - x_0}$$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{y_1 - y_0}{x_1 - x_0}\right)^2} dx = \sqrt{1 + \left(\frac{y_1 - y_0}{x_1 - x_0}\right)^2} \cdot \int_{x_0}^{x_1} dx$$

$$= \sqrt{1 + \left(\frac{y_1 - y_0}{x_1 - x_0}\right)^2} \cdot (x_1 - x_0) = \sqrt{1 + \left(\frac{y_1 - y_0}{x_1 - x_0}\right)^2} \cdot \sqrt{(x_1 - x_0)^2}$$

$$= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$ <u>ok</u> , recall : $\sqrt{a}\sqrt{b} = \sqrt{ab}$

<u>example</u> : circumference of a circle of radius r , $L = 2\pi r$

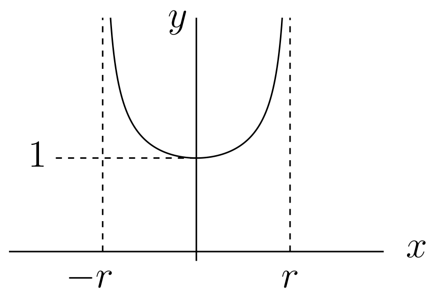


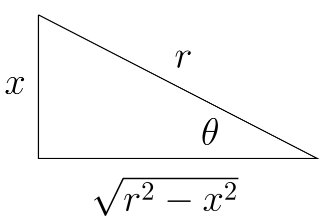
$$L = 2\int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

$$x^2 + y^2 = r^2 \ \Rightarrow\ f(x) = (r^2 - x^2)^{1/2}\ ,\ f'(x) = \frac{1}{2}(r^2 - x^2)^{-1/2}\cdot(-2x) = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$1 + (f'(x))^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2 + x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$$

$$L \ = \ 2\int_a^b \sqrt{1 + (f'(x))^2}\,dx \ = \ 2\int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\,dx \ :\ \text{improper}$$

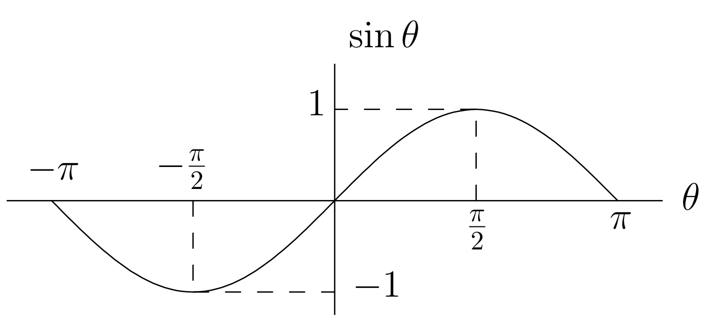




$$\sin\theta = \frac{x}{r} \ \Rightarrow \cos\theta\,d\theta = \frac{dx}{r}\ ,\ \ \sec\theta = \frac{r}{\sqrt{r^2 - x^2}}$$

$$L = 2\int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\,dx = 2\int_{-\pi/2}^{\pi/2} \cancel{\sec\theta}\cdot r\,\cancel{\cos\theta}\,d\theta = 2r\int_{-\pi/2}^{\pi/2} d\theta = 2r\cdot\theta\Big|_{-\pi/2}^{\pi/2}$$

$$= 2r\cdot\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 2\pi r \qquad \underline{\text{ok}}$$



$$x = r \Rightarrow \sin\theta = 1 \Rightarrow \theta = \tfrac{\pi}{2}$$

$$x = -r \Rightarrow \sin\theta = -1 \Rightarrow \theta = -\tfrac{\pi}{2}$$

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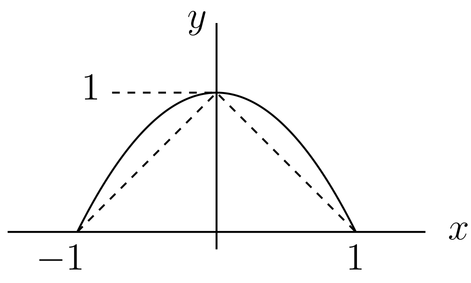
<u>example</u> : parabola , $y = 1 - x^2$, $-1 \le x \le 1$

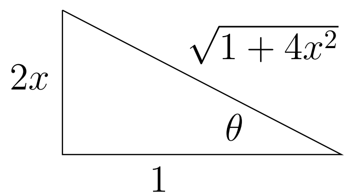


$$L > 2\sqrt{2} = 2.8284$$

$$f(x) = 1 - x^2 \Rightarrow f'(x) = -2x$$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_{-1}^1 \sqrt{1 + 4x^2}\, dx = 2\int_0^1 \sqrt{1 + 4x^2}\, dx$$



$$\tan\theta = 2x \;\Rightarrow\; \sec^2\theta d\theta = 2dx$$

$$\sec\theta = \sqrt{1 + 4x^2}$$

$$\int \sqrt{1 + 4x^2}\, dx = \int \sec\theta \cdot \tfrac{1}{2}\sec^2\theta d\theta = \tfrac{1}{2}\int \sec^3\theta d\theta$$

$$\int \sec^3\theta d\theta = \int \frac{d\theta}{\cos^3\theta} = \int \frac{\cos\theta}{\cos^4\theta} d\theta = \int \frac{\cos\theta d\theta}{(1 - \sin^2\theta)^2} = \int \frac{du}{(1 - u^2)^2}$$

$$u = \sin\theta \;\Rightarrow\; du = \cos\theta d\theta$$

$$(1 - u^2)^2 = ((1 + u)(1 - u))^2 = (1 + u)^2(1 - u)^2$$

$$\frac{1}{(1 - u^2)^2} = \frac{a}{1 + u} + \frac{b}{(1 + u)^2} + \frac{c}{1 - u} + \frac{d}{(1 - u)^2} = \cdots$$

<u>alternative</u>

$$\int \sec^3\theta d\theta = \int \sec\theta \cdot \sec^2\theta d\theta$$

$$u = \sec\theta\,,\ dv = \sec^2\theta d\theta \;\Rightarrow\; du = \sec\theta\tan\theta d\theta\,,\ v = \tan\theta$$

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \tan\theta \cdot \sec\theta\tan\theta d\theta = \sec\theta\tan\theta - \int \sec\theta \cdot \frac{\sin^2\theta}{\cos^2\theta} d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta(1 - \cos^2\theta)d\theta = \sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta$$

$$2\int \sec^3\theta d\theta = \sec\theta\tan\theta + \int \sec\theta d\theta$$

$$\int \sec^3\theta d\theta = \tfrac{1}{2}\left(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)\right)$$

$$\int \sqrt{1 + 4x^2}\, dx = \tfrac{1}{2} \cdot \tfrac{1}{2}\left(2x\sqrt{1 + 4x^2} + \ln\!\left(2x + \sqrt{1 + 4x^2}\right)\right)$$

$$L = 2\int_0^1 \sqrt{1 + 4x^2}\, dx = \tfrac{1}{2}\left(2x\sqrt{1 + 4x^2} + \ln\!\left(2x + \sqrt{1 + 4x^2}\right)\right)\Big|_0^1$$

$$= \tfrac{1}{2}(2\sqrt{5} + \ln(2 + \sqrt{5}\,)) = 2.9579$$

1.8 surface area

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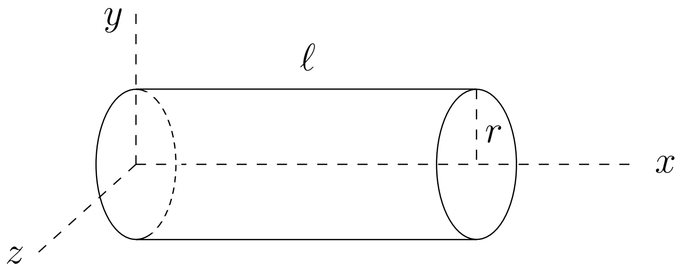
problem : find the area of the surface formed by rotating a curve about an axis

<u>example</u> : cylinder , $S = 2\pi r\ell$

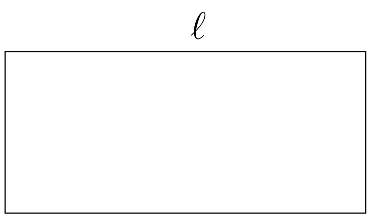


ℓ : length

r : radius

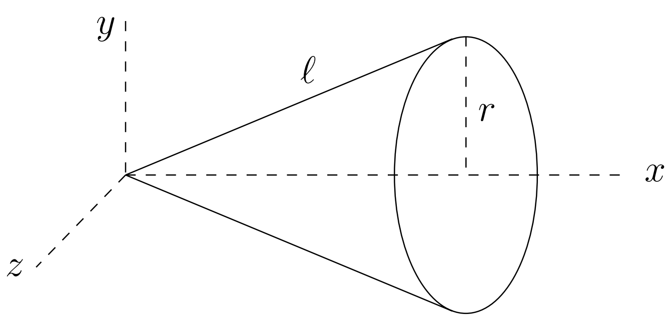
S : surface area

To find S, cut the cylinder and spread it flat to form a rectangle.



$2\pi r \;\Rightarrow\; S = 2\pi r\ell$ <u>ok</u>

<u>example</u> : cone , $S = \pi r\ell$

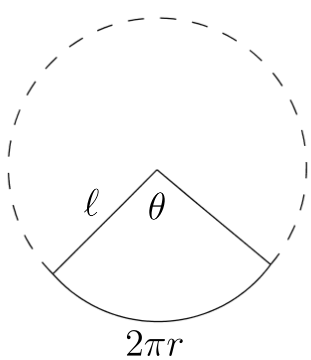


ℓ : slant length of cone

r : radius

S : surface area

To find S, cut the cone and spread it flat to form a circular sector (slice of pie).



θ : sector angle

ℓ : radius of circle

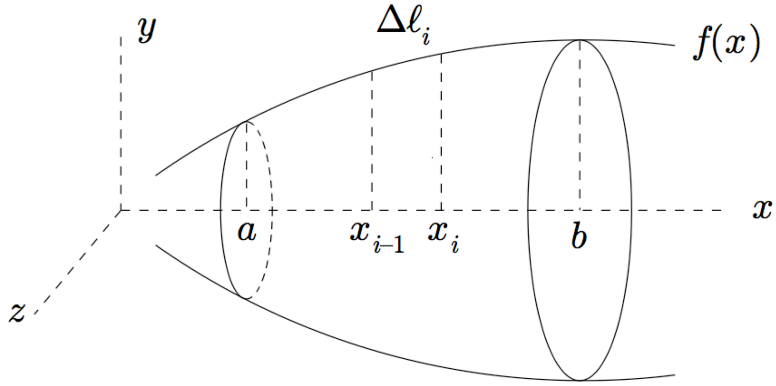
$2\pi r$: length of sector edge

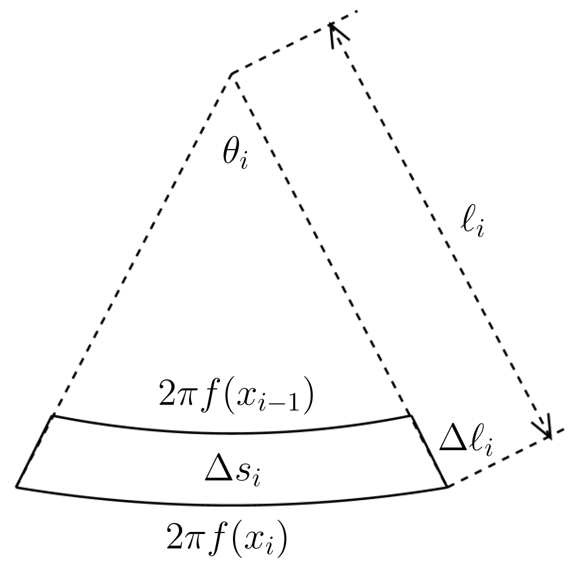
S : area of sector = surface area of cone

1. $\dfrac{2\pi r}{2\pi\ell} = \dfrac{\theta}{2\pi} \Rightarrow 2\pi r = \cancel{2\pi}\ell \cdot \dfrac{\theta}{\cancel{2\pi}} = \ell\theta$

2. $\dfrac{S}{\pi\ell^2} = \dfrac{\theta}{2\pi} \Rightarrow S = \cancel{\pi}\ell^2 \cdot \dfrac{\theta}{2\cancel{\pi}} = \frac{1}{2}\ell^2\theta = \frac{1}{2}\ell \cdot \ell\theta = \frac{1}{2}\ell \cdot 2\pi r = \pi r\ell$ <u>ok</u>

general case





$\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$

$\Delta\ell_i$: slant length of ith slice , Δs_i : surface area of ith slice

$$\Delta s_i \approx \tfrac{1}{2}\ell_i^2\theta_i - \tfrac{1}{2}(\ell_i - \Delta\ell_i)^2\theta_i = \tfrac{1}{2}\cancel{\ell_i^2}\theta_i - \tfrac{1}{2}(\cancel{\ell_i^2} - 2\ell_i\Delta\ell_i + \cancel{\Delta\ell_i^2})\theta_i = \ell_i\Delta\ell_i\theta_i$$
$$= \ell_i\theta_i\Delta\ell_i \approx 2\pi f(x_i)\Delta\ell_i$$

recall : $\Delta\ell_i^2 = \Delta x^2 + \Delta y_i^2 \Rightarrow \Delta\ell_i \approx \sqrt{1+(f'(x_i))^2}\,\Delta x$

$$S = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta s_i = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i)\sqrt{1+(f'(x_i))^2}\,\Delta x = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

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check 1 : cylinder , $S = 2\pi r\ell$

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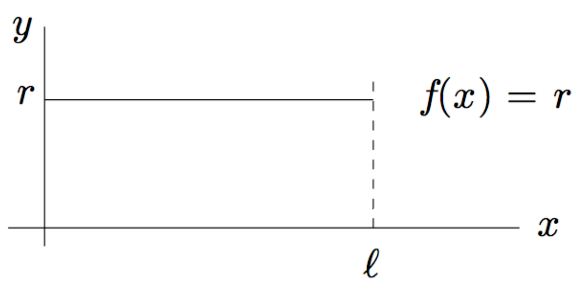
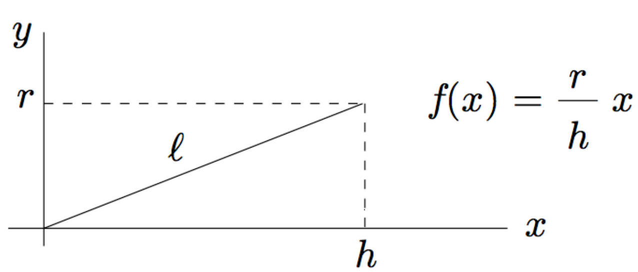


$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$
$$= \int_0^\ell 2\pi r dx = 2\pi r\int_0^\ell dx = 2\pi r\ell \quad \underline{\text{ok}}$$

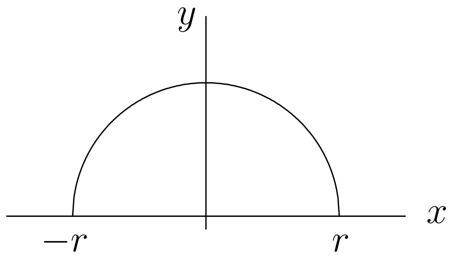
---

check 2 : cone , $S = \pi r\ell$



$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx = \int_0^h 2\pi\frac{r}{h}x\sqrt{1+\frac{r^2}{h^2}}\,dx = 2\pi\frac{r}{h}\sqrt{1+\frac{r^2}{h^2}}\cdot\int_0^h x dx$$
$$= 2\pi\frac{r}{h}\sqrt{1+\frac{r^2}{h^2}}\cdot\frac{h^2}{2} = \pi r\sqrt{1+\frac{r^2}{h^2}}\cdot h = \pi r\sqrt{h^2+r^2} = \pi r\ell \quad \underline{\text{ok}}$$

<u>example</u>: sphere , $S = 4\pi r^2$



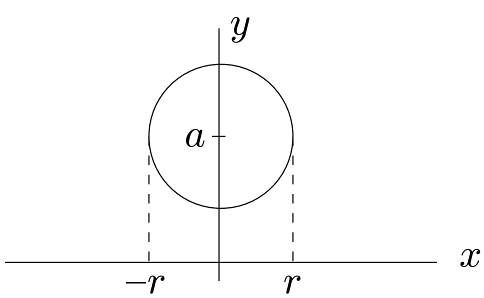
$$f(x) = (r^2 - x^2)^{1/2} \Rightarrow f'(x) = \tfrac{1}{2}(r^2 - x^2)^{-1/2} \cdot -2x = \frac{-x}{\sqrt{r^2 - x^2}}$$

$$1 + (f'(x))^2 = 1 + \frac{x^2}{r^2 - x^2} = \frac{r^2 - x^2 + x^2}{r^2 - x^2} = \frac{r^2}{r^2 - x^2}$$

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx = 2\pi \int_{-r}^{r} \cancel{\sqrt{r^2 - x^2}} \cdot \frac{r}{\cancel{\sqrt{r^2 - x^2}}}\,dx$$

$$= 2\pi r \int_{-r}^{r} dx = 2\pi r \cdot 2r = 4\pi r^2 \quad \underline{\text{ok}}$$

<u>example</u> : torus , $S = 4\pi^2 ar$



assume $a > r$

equation of circle : $x^2 + (y - a)^2 = r^2 \Rightarrow y = a \pm \sqrt{r^2 - x^2}$

upper semicircle : $f_+(x) = a + \sqrt{r^2 - x^2}$

lower semicircle : $f_-(x) = a - \sqrt{r^2 - x^2}$

$$S = S_+ + S_- = \int_{-r}^{r} 2\pi f_+(x)\sqrt{1 + (f_+'(x))^2}\,dx + \int_{-r}^{r} 2\pi f_-(x)\sqrt{1 + (f_-'(x))^2}\,dx$$

$$1 + (f_+'(x))^2 = \frac{r^2}{r^2 - x^2} = 1 + (f_-'(x))^2$$

$$S = 2\pi \int_{-r}^{r} (a + \cancel{\sqrt{r^2 - x^2}} + a - \cancel{\sqrt{r^2 - x^2}}) \frac{r}{\sqrt{r^2 - x^2}}\,dx$$

$$= 2\pi \cdot 2a \cdot \underbrace{\int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\,dx}_{\uparrow} = 2\pi \cdot 2a \cdot \pi r = 4\pi^2 ar \quad \underline{\text{ok}}$$

arclength of a semicircle (page 22)

<u>note</u>

1. $S_{\text{torus}} = 4\pi^2 ar = 2\pi a \cdot 2\pi r =$ product of two circles

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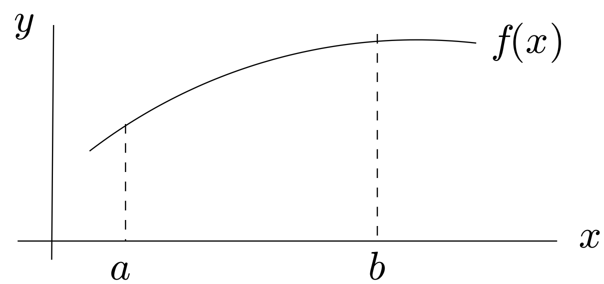
2. For the cylinder and cone, we can find S by cutting the surface and spreading it flat, but this does not work for the sphere and torus. (differential geometry, Math 433)

<u>summary</u>



area under graph of $y = f(x)$ for $a \le x \le b$: $A = \int_a^b f(x)dx$

arclength $\cdots\cdots$: $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

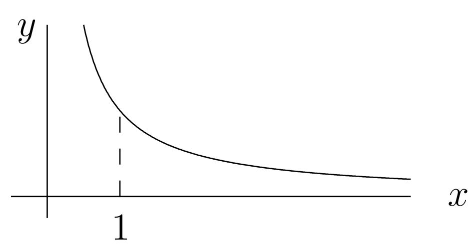
surface area $\cdots\cdots$: $S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$

volume $\cdots\cdots$: $V = \int_a^b \pi f(x)^2 dx$

<u>example</u>

$f(x) = \dfrac{1}{x}$, $1 \le x < \infty$

$A = \int_1^\infty \dfrac{dx}{x}$: diverges, $p = 1$



$L = \int_1^\infty \sqrt{1 + \dfrac{1}{x^4}}\, dx$: diverges , comparison test , $\sqrt{1 + \dfrac{1}{x^4}} \ge 1$

$S = \int_1^\infty 2\pi \dfrac{1}{x}\sqrt{1 + \dfrac{1}{x^4}}\, dx$: diverges , comparison test , $\dfrac{1}{x}\sqrt{1 + \dfrac{1}{x^4}} \ge \dfrac{1}{x}$

$V = \int_1^\infty \dfrac{\pi}{x^2} dx = -\dfrac{\pi}{x}\Big|_1^\infty = \pi$: converges , $p = 2$

This shape is called <u>Gabriel's horn</u>; it has finite volume, but infinite surface area.

1.9 center of mass

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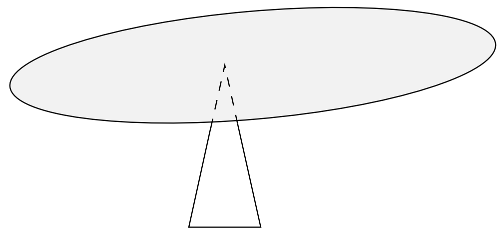
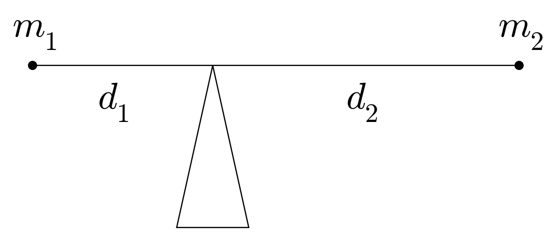
problem : Find the point at which a thin plate balances.

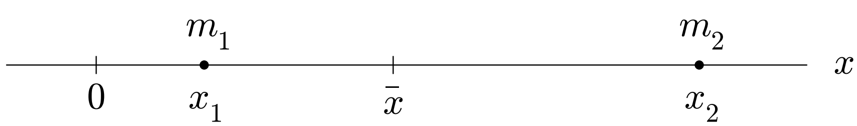


<u>example</u> : 2 point masses m_1, m_2 connected by a rod of negligible mass



balance of torque : $m_1 g \cdot d_1 = m_2 g \cdot d_2$ (prevents tipping)

$\Rightarrow\ m_1 d_1 = m_2 d_2$: balance of moments



x_1 , x_2 , $\bar{x}$: coordinates of m_1 , m_2 , CM (center of mass)

$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}) \ \Rightarrow\ (m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2 \ \Rightarrow\ \bar{x} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

$m_i x_i$: <u>moment</u> of mass i about $x = 0$, units are mass $\times$ distance

The balance of moments can also be written as $m_1(\bar{x} - x_1) + m_2(\bar{x} - x_2) = 0$.

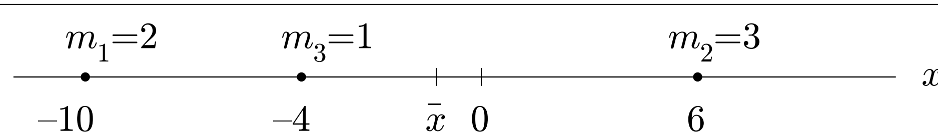
<u>example</u> : n point masses $m_1, \ldots, m_n$ connected by a rod of negligible mass

balance of moments $\Rightarrow\ \sum_{i=1}^{n} m_i(\bar{x} - x_i) = 0 \ \Rightarrow\ \sum_{i=1}^{n} m_i \bar{x} = \sum_{i=1}^{n} m_i x_i$

$$\Rightarrow\ \bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{M}{m}\ ,\ M = \sum_{i=1}^{n} m_i x_i : \text{total moment}\ ,\ m = \sum_{i=1}^{n} m_i : \text{total mass}$$

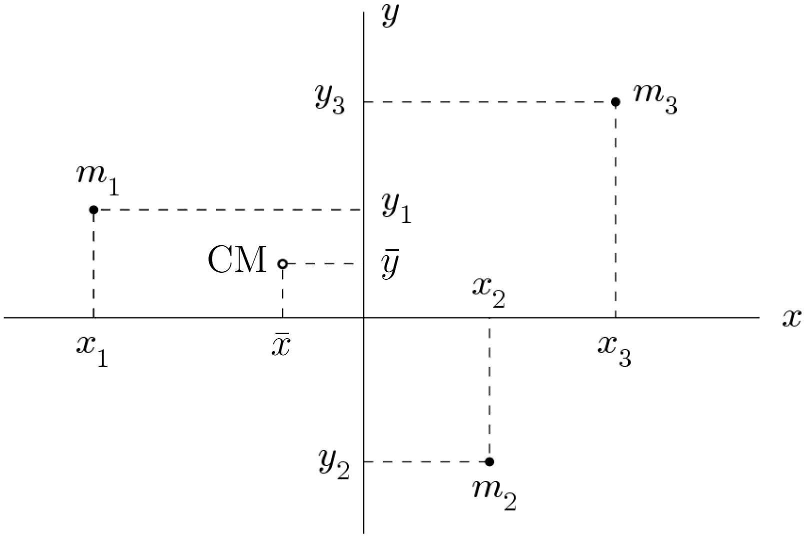
If $n = 2$, this agrees with the previous formula for $\bar{x}$.

<u>example</u> :



$$\left.\begin{array}{l} M = m_1 x_1 + m_2 x_2 + m_3 x_3 = -20 + 18 - 4 = -6 \\ m = m_1 + m_2 + m_3 = 2 + 3 + 1 = 6 \end{array}\right\} \Rightarrow\ \bar{x} = \frac{M}{m} = \frac{-6}{6} = -1$$

<u>two</u>-dimensional case



balance of moments : $\sum_{i=1}^{n} m_i(\bar{x}-x_i)=0$, $\sum_{i=1}^{n} m_i(\bar{y}-y_i)=0$

$\Rightarrow \bar{x} = \dfrac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \dfrac{M_y}{m}$, M_y : <u>total moment about $y-$axis</u> , $M_y = m\bar{x}$

$\bar{y} = \dfrac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} = \dfrac{M_x}{m}$, M_x : <u>total moment about $x-$axis</u> , $M_x = m\bar{y}$

1. M_y, M_x are the same as if the n point masses are concentrated into a single point mass located at CM $= (\bar{x}, \bar{y})$
2. $\bar{x}, \bar{y}$ are weighted averages of $\{x_i\}, \{y_i\}$

<u>example</u> : $m_1 = m_2 = m_3 = 1$

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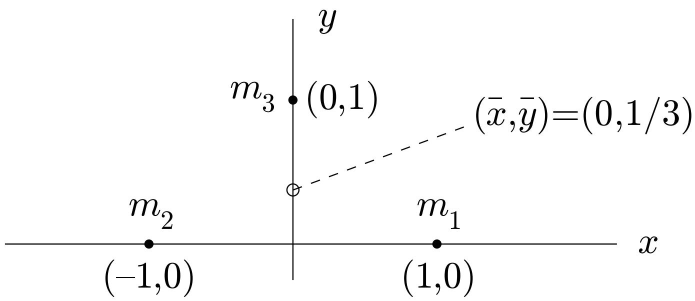
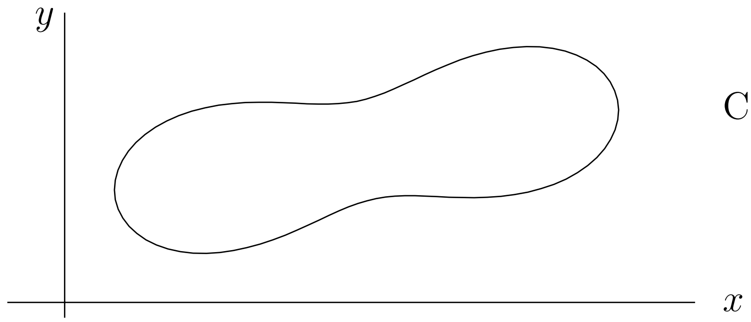


$$\bar{x} = \frac{M_y}{m} = \frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3} = \frac{1-1+0}{3} = 0$$

$$\bar{y} = \frac{M_x}{m} = \frac{m_1y_1+m_2y_2+m_3y_3}{m_1+m_2+m_3} = \frac{0+0+1}{3} = \frac{1}{3}$$

continuous mass distribution

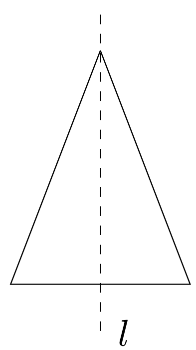
Consider a region of uniform density ρ in the xy-plane.



$\text{CM} = (\bar{x}, \bar{y}) = \,?$

symmetry principle : If a region is symmetric about a line l, then CM lies on l.

example : isoceles triangle , CM lies on l



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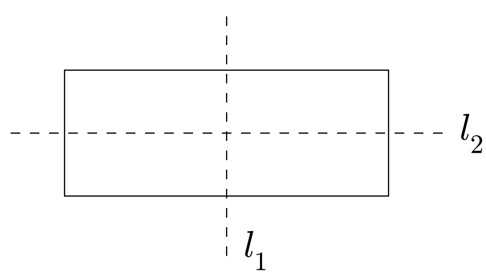
example : rectangle , CM lies on l_1 and $l_2 \Rightarrow$ CM is at center of rectangle



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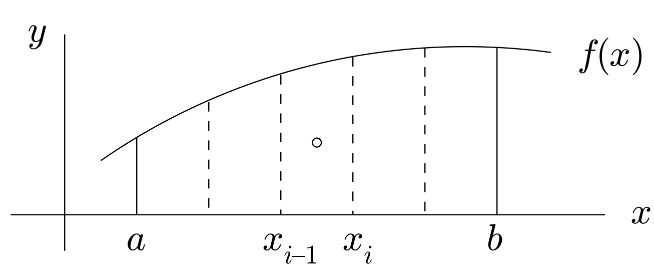
case 1 : $R = \{(x, y) : a \le x \le b\,,\ 0 \le y \le f(x)\}$



R_i : ith rectangle

CM of $R_i = (x_i^*, \frac{1}{2} f(x_i^*))$

balance of moments $\Rightarrow\ \bar{x} = \dfrac{M_y}{m}\,,\ \bar{y} = \dfrac{M_x}{m}$, but how do we find M_x , M_y , m?

$\Delta x = \dfrac{b-a}{n}\ ,\ x_i = a + i\Delta x\ ,\ x_i^* = \frac{1}{2}(x_{i-1} + x_i)$

mass of R_i = density $\times$ area = $\rho \cdot f(x_i^*)\Delta x$

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho f(x_i^*)\Delta x = \int_a^b \rho f(x)dx$$

<u>moment principle 1</u> : The moments of a rectangle are the same as if all the mass is concentrated at CM.

moment of R_i about $y-$axis = mass $\times$ distance = $\rho f(x_i^*)\Delta x \cdot x_i^*$

moment ..... ”..... $x-$axis = ...”....... = $\rho f(x_i^*)\Delta x \cdot \frac{1}{2} f(x_i^*)$

<u>moment principle 2</u> : The moment of a union of rectangles is the sum of the moments of the rectangles.

$$M(R_1 \cup R_2 \cdots \cup R_n) = \sum_{i=1}^{n} M(R_i)$$

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho x_i^* f(x_i^*)\Delta x = \int_a^b \rho x f(x)dx$$

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \tfrac{1}{2} f(x_i^*)^2 \Delta x = \tfrac{1}{2}\int_a^b \rho f(x)^2 dx$$

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b \cancel{\rho} x f(x)dx}{\int_a^b \cancel{\rho} f(x)dx}, \ \bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b \cancel{\rho} f(x)^2 dx}{\int_a^b \cancel{\rho} f(x)dx}$$, assume $\rho = 1$ from now on

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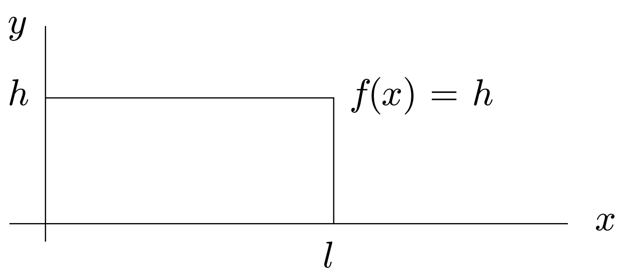
<u>example</u> : rectangle



$$M_y = \int_a^b x f(x)dx = \int_0^l x \cdot h dx = h \frac{x^2}{2}\Big|_0^l = \frac{hl^2}{2}$$

$$M_x = \tfrac{1}{2}\int_a^b f(x)^2 dx = \tfrac{1}{2}\int_0^l h^2 dx = \frac{h^2 l}{2}$$

$$m = \int_a^b f(x)dx = \int_0^l h dx = hl$$

$$\bar{x} = \frac{M_y}{m} = \frac{hl^2}{2} \cdot \frac{1}{hl} = \frac{l}{2}, \ \bar{y} = \frac{M_x}{m} = \frac{h^2 l}{2} \cdot \frac{1}{hl} = \frac{h}{2}$$ <u>ok</u>

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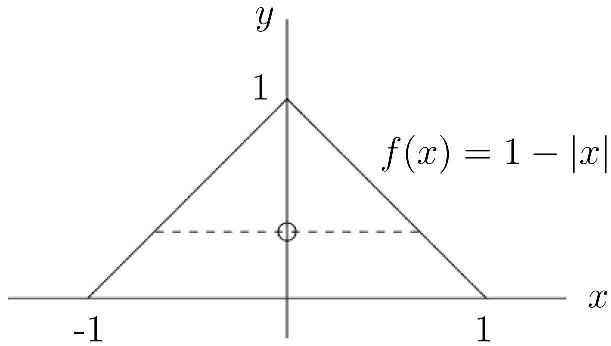
<u>example</u> : triangular plate



$$f(x) = 1 - |x| = \begin{cases} 1 - x \text{ if } x \geq 0 \\ 1 + x \text{ if } x \leq 0 \end{cases}$$

symmetry $\Rightarrow \ \bar{x} = 0$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b f(x)^2 dx}{\int_a^b f(x) dx} = \frac{\frac{1}{2}\int_{-1}^{0}(1+x)^2 dx \ + \ \frac{1}{2}\int_0^1 (1-x)^2 dx}{\frac{1}{2} \cdot 2 \cdot 1}$$

$$= \frac{1}{2} \cdot \frac{1}{3}(1+x)^3\Big|_{-1}^{0} + \frac{1}{2} \cdot -\frac{1}{3}(1-x)^3\Big|_0^1 = \frac{1}{6}(1-0) - \frac{1}{6}(0-1) = \frac{1}{3}$$

$\Rightarrow$ CM $= (0, \frac{1}{3})$

question :

The line $y = \bar{y}$ divides the triangle into 2 parts; do they have the same area?

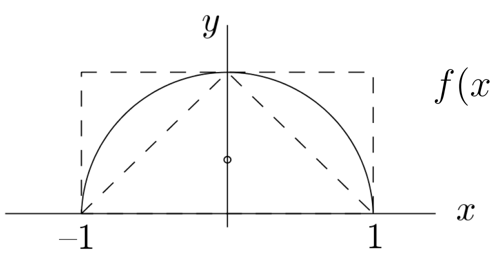
area of upper part $= \frac{1}{2} \cdot \frac{4}{3} \cdot \frac{2}{3} = \frac{4}{9}$

area of lower part $= \frac{1}{3} \cdot \frac{1}{2} \cdot (2 + \frac{4}{3}) = \frac{1}{6} \cdot \frac{10}{3} = \frac{5}{9}$

check : total area $= 1$ <u>ok</u>

Even though the CM lies on the boundary between the 2 parts, the lower part has larger area; this is due to the balance of moments (and moment = mass $\times$ distance).

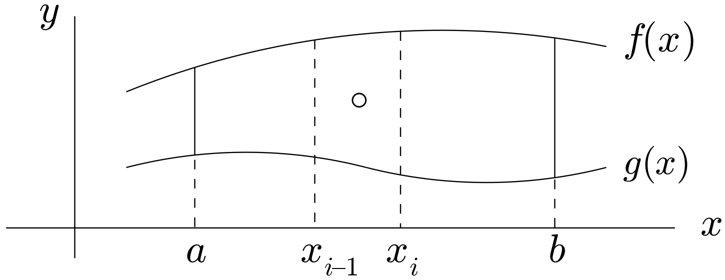
<u>example</u> : half-disk



$f(x) = \sqrt{1 - x^2}$ we expect $\frac{1}{3} < \bar{y} < \frac{1}{2}$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b f(x)^2 dx}{\int_a^b f(x) dx} = \frac{\int_0^1 (1-x^2) dx}{\pi/2} = \frac{\left(x - \frac{1}{3}x^3\right)\Big|_0^1}{\pi/2} = \frac{\frac{2}{3}}{\pi/2} = \frac{4}{3\pi} = 0.4244$$

<u>case 2</u> : $R = \{(x,y) : a \le x \le b\,,\ g(x) \le y \le f(x)\}$



mass of ith rectangle $\approx (f(x_i^*) - g(x_i^*))\Delta x$

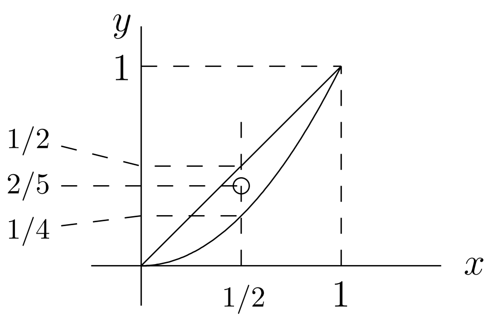
(recall : moment = mass $\times$ distance)

moment $\cdots$" $\cdots$ about y-axis $\approx (f(x_i^*) - g(x_i^*))\Delta x \cdot x_i^*$

moment $\cdots$" $\cdots$ about x-axis $\approx (f(x_i^*) - g(x_i^*))\Delta x \cdot \frac{1}{2}(f(x_i^*) + g(x_i^*))$

$$\bar{x} = \frac{M_y}{m} = \frac{\int_a^b x(f(x) - g(x))dx}{\int_a^b (f(x) - g(x))dx}\ ,\ \bar{y} = \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b (f(x)^2 - g(x)^2)dx}{\int_a^b (f(x) - g(x))dx}$$

<u>example</u> : $f(x) = x\ ,\ g(x) = x^2\ ,\ 0 \le x \le 1$ | 21 Fri 10/4



symmetry? no

$$m = \int_a^b (f(x) - g(x))dx = \int_0^1 (x - x^2)dx = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}$$

$$M_x = \tfrac{1}{2}\int_a^b (f(x)^2 - g(x)^2)dx = \tfrac{1}{2}\int_0^1 (x^2 - x^4)dx = \tfrac{1}{2}\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{1}{15}$$

$$M_y = \int_a^b x(f(x) - g(x))dx = \int_0^1 (x^2 - x^3)dx = \tfrac{1}{3} - \tfrac{1}{4} = \tfrac{1}{12}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}\ ,\ \bar{y} = \frac{M_x}{m} = \frac{\frac{1}{15}}{\frac{1}{6}} = \frac{2}{5}\ \Rightarrow$$ CM is closer to top edge

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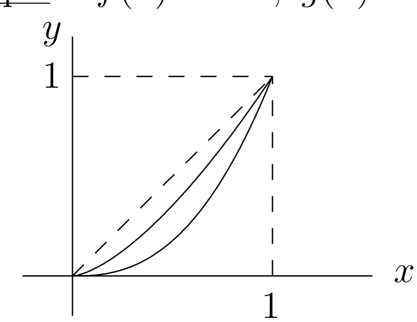
<u>example</u> : $f(x) = x^m\,,\ g(x) = x^n$



For some choice of m and n,
the CM lies <u>outside</u> the region. (hw7)

1.10 probability

X : random variable

examples

X = velocity of a gas molecule

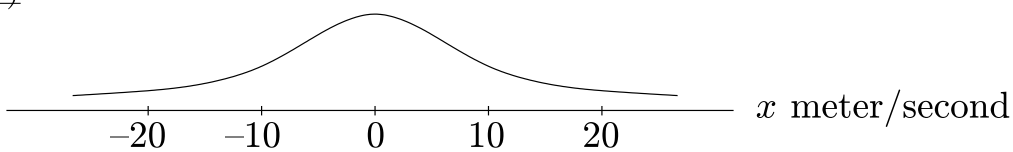
X = waiting time in the supermarket checkout line

X = GPA of a college student

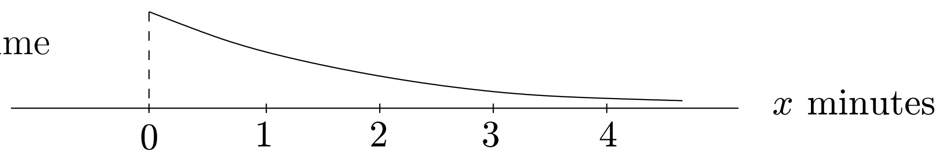
definition : A random variable X has a probability density function $f(x)$ such that $\int_a^b f(x)dx$ = probability that X lies between a and b = prob$(a \le X \le b)$.

examples of $f(x)$

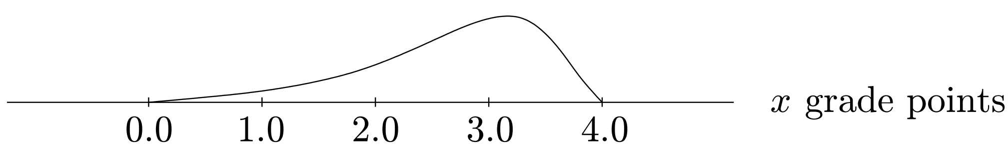
X = velocity



X = waiting time



X = GPA



large $f(x) \Leftrightarrow$ high probability that X is near x

small $f(x) \Leftrightarrow$ low ·········· " ···············

note : probability density is similar to mass density or charge density

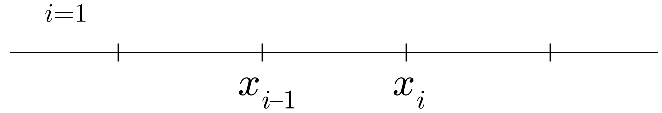
properties of a pdf

1. $f(x) \ge 0$

2. $\int_{-\infty}^{\infty} f(x)dx \;=\; \text{prob}(-\infty < X < \infty) \;=\; 1$

definition : The mean value of a random variable X (average value, expected value) is denoted by $\mu = \mu(X)$ and is defined by

$$\mu = \int_{-\infty}^{\infty} x f(x)dx \approx \sum_{i=1}^{n} x_i^* f(x_i^*)\Delta x \approx \sum_{i=1}^{n} x_i^* \cdot \text{prob}(x_{i-1} \le X \le x_i),$$



i.e. the values of x_i^* are weighted by the probability that X is near x_i^*.

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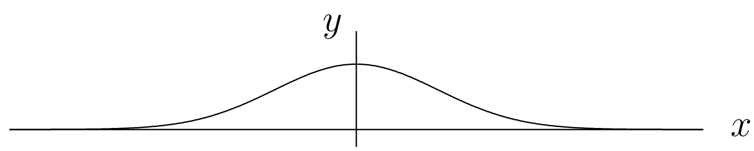
<u>example</u> : Gaussian pdf

$$f(x) = \frac{1}{\sqrt{\pi}}\, e^{-x^2}$$



1. $f(x) \geq 0$

2. $\displaystyle\int_{-\infty}^{\infty} f(x)dx \;=\; \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}}\, e^{-x^2} dx \;=\; \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx \;=\; \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} \;=\; 1$

$\uparrow$ Math 215, 285

3. $\displaystyle\mu = \int_{-\infty}^{\infty} x f(x) dx \;=\; \int_{-\infty}^{\infty} x\, \frac{1}{\sqrt{\pi}}\, e^{-x^2} dx \;=\; 0$

<u>definition</u> : The <u>median value</u> of a random variable X is denoted by $m = m(X)$ and is defined by $\text{prob}(X \leq m) = \text{prob}(X \geq m) = \frac{1}{2}$.

1. This is equivalent to $\int_{-\infty}^{m} f(x)dx = \int_{m}^{\infty} f(x)dx = \frac{1}{2}$, i.e. half the area under the graph of $f(x)$ lies to the left of m and half lies to the right.

2. For the Gaussian pdf we have $m = \mu$, but in general $m \neq \mu$. (more later)

<u>definition</u> : The <u>standard deviation</u> of a random variable X is denoted by $\sigma = \sigma(X)$ and is defined by $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$, where σ^2 is the <u>variance</u>.

small $\sigma \Leftrightarrow X$ is more likely to be near μ

large $\sigma \Leftrightarrow X$ is less likely to be near μ

<u>example</u>

$$f(x) = \frac{1}{\sqrt{\pi}}\, e^{-x^2} \;\Rightarrow\; \mu = 0 \;,\; \sigma = \,?$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx \;=\; \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{\pi}}\, e^{-x^2} dx \quad , \quad u = x \;,\; dv = x e^{-x^2} dx$$

$$du = dx \;,\; v = \frac{e^{-x^2}}{-2}$$

$$= \frac{1}{\sqrt{\pi}} \left(\cancel{x \cdot \frac{e^{-x^2}}{-2} \Big|_{-\infty}^{\infty}} + \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} dx \right) = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \frac{1}{2}$$

$$\Rightarrow \sigma = \frac{1}{\sqrt{2}} \;=\; 0.7071$$

<u>definition</u> : Let μ and $\sigma > 0$ be given and define $f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}}\, e^{\dfrac{-(x-\mu)^2}{2\sigma^2}}$.

Then $f(x)$ is the pdf of a random variable X called a <u>normal distribution</u> with mean μ and standard deviation σ.

1. The Gaussian pdf corresponds to $\mu = 0$, $\sigma = 1/\sqrt{2}$.
2. μ <u>shifts</u> $f(x)$ along the x-axis and σ <u>stretches</u> the height and width of $f(x)$

<u>check</u> (hw7)

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2} dx = 1$$

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} x\, \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2} dx = \mu$$

$$\int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2} dx = \sigma^2$$

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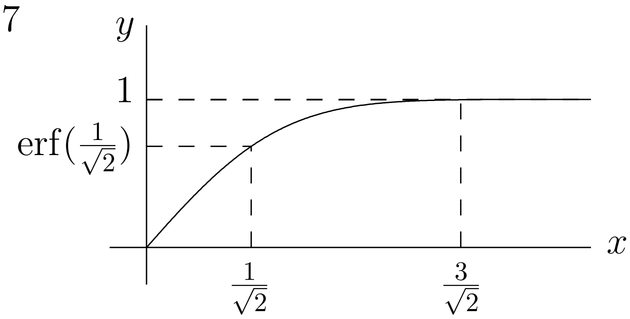
1. Find the probability that X is within 1 sd of μ.

$$\text{prob}(\,\mu-\sigma \le X \le \mu+\sigma) = \int_{\mu-\sigma}^{\mu+\sigma} f(x)dx = \int_{\mu-\sigma}^{\mu+\sigma} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx \;\;,\;\; u = \frac{x-\mu}{\sqrt{2\sigma^2}}$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-u^2} \cdot \sqrt{2\sigma^2}\, du = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{\pi}}\, e^{-u^2} du \;:\; \text{Gaussian pdf} \qquad du = \frac{dx}{\sqrt{2\sigma^2}}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{2}}} e^{-t^2} dt \;=\; \text{erf}(\tfrac{1}{\sqrt{2}}) = 0.6827$$

$$\text{recall : erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$



2. Find the probability that X is 3 sd or more greater than μ.

$$\text{prob}(X \ge \mu + 3\sigma) \;=\; \int_{\mu+3\sigma}^{\infty} f(x)dx \;=\; \cdots \;=\; \int_{\frac{3}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{\pi}}\, e^{-x^2} dx$$

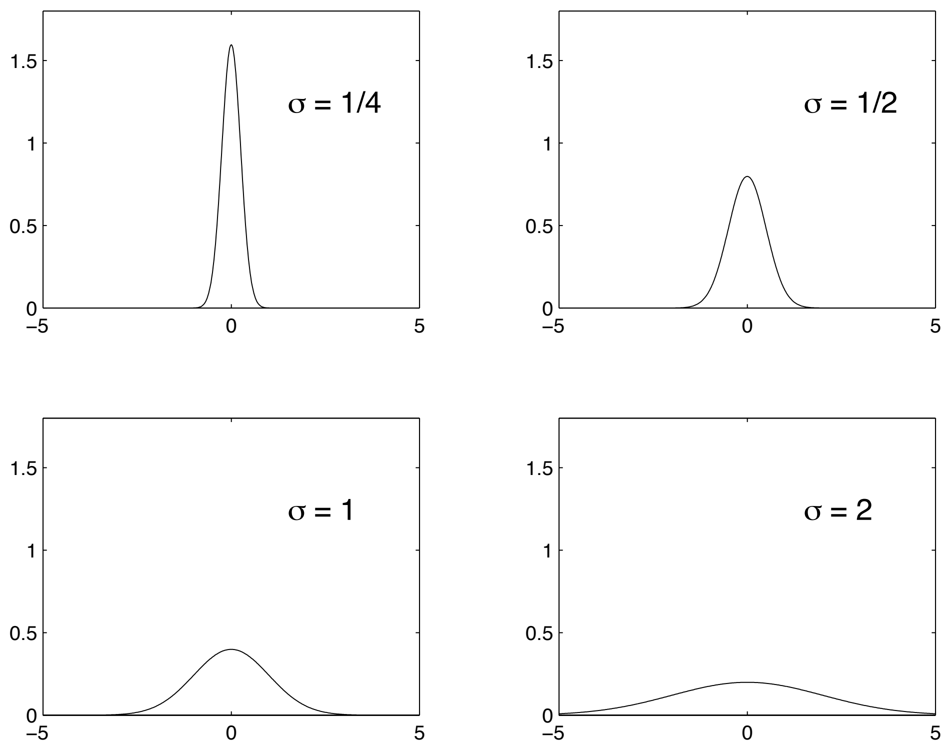
$$= \frac{1}{\sqrt{\pi}} \left(\int_0^{\infty} e^{-x^2} dx - \int_0^{\frac{3}{\sqrt{2}}} e^{-x^2} dx \right) = \tfrac{1}{2}\left(\text{erf}(\infty) - \text{erf}(\tfrac{3}{\sqrt{2}})\right) = 0.001349$$

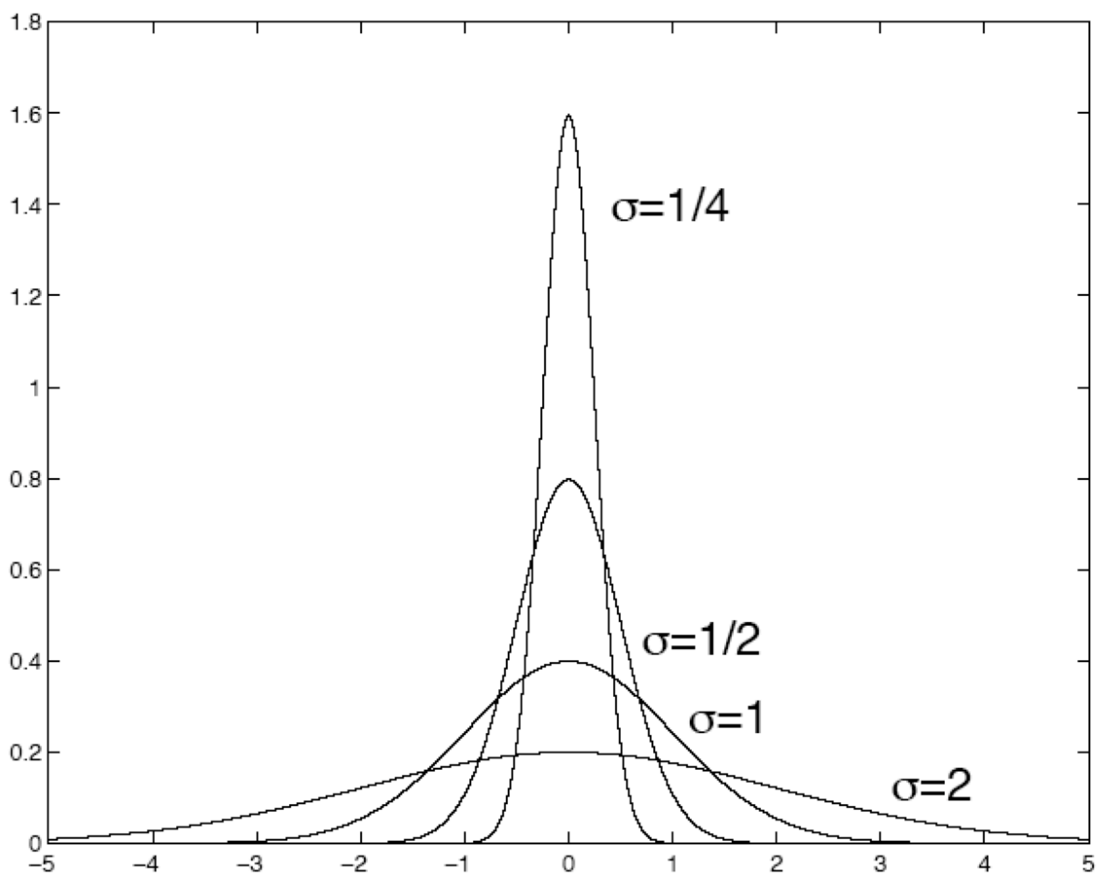
<u>example</u>

Annual rainfall in Michigan is normally distributed with $\mu = 34$ in, $\sigma = 4$ in.

1. 68% of years have rainfall between 30 in and 38 in (twice in 3 years)
2. 0.13% . . . . ” . . . . greater than or equal to 46 in (once in 770 years)

pdf of a normal distribution : $f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}$, $\mu = 0$, $\sigma = \frac{1}{4}, \frac{1}{2}, 1, 2$



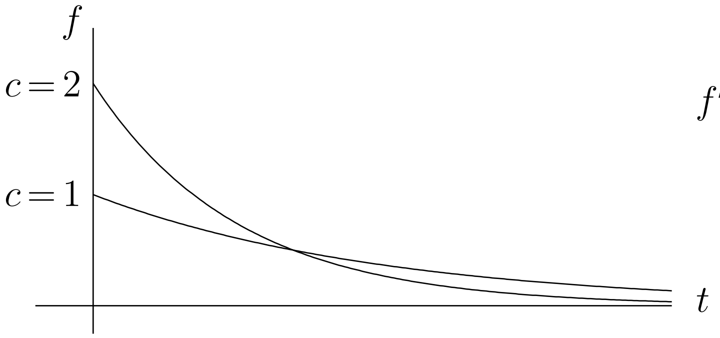


These graphs also describe how a sugar cube dissolves in a cup of hot coffee.

<u>exponential distribution</u>

T = waiting time in the supermarket checkout line (minutes)

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ c\,e^{-ct} & \text{if } t \geq 0 \end{cases} \quad , \quad c > 0$$



$$f'(t) = -c^2 e^{-ct} \Rightarrow f'(0) = -c^2$$

<u>check</u>

$$\int_{-\infty}^{\infty} f(t)dt = \int_{-\infty}^{0} \cancel{f(t)}dt + \int_0^{\infty} f(t)dt = \int_0^{\infty} c\,e^{-ct}dt = -e^{-ct}\Big|_0^{\infty} = 0 + 1 = 1 \quad \underline{\text{ok}}$$

$$\text{average waiting time} = \mu = \int_{-\infty}^{\infty} t f(t)dt = \int_{-\infty}^{0} t\cancel{f(t)}dt + \int_0^{\infty} t f(t)dt = \int_0^{\infty} tce^{-ct}dt$$

$$= \int_0^{\infty} u\,e^{-u}\frac{du}{c} = \frac{1}{c} \qquad u = ct\,,\ du = cdt$$

<u>example</u>: Assume the average waiting time is $\mu = 5$ minutes; then $c = \frac{1}{5}$. 24 Wed 10/9

1. Find the probability that a shopper waits 1 minute or less.

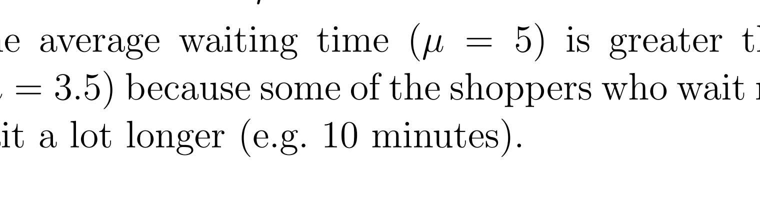
$$\text{prob}(0 \leq T \leq 1) = \int_0^1 f(t)\,dt = \int_0^1 c\,e^{-ct}\,dt = -e^{-ct}\Big|_0^1 = -e^{-c} + 1 = 1 - e^{-\frac{1}{5}} = 0.1813$$

Hence 18% of shoppers wait 1 minute or less.

2. Find the probability that a shopper waits 5 minutes or more.

$$\text{prob}(T \geq 5) = \int_5^{\infty} f(t)\,dt = \int_5^{\infty} c\,e^{-ct}\,dt = -e^{-ct}\Big|_5^{\infty} = 0 - (-e^{-5c}) = e^{-1} = 0.3679$$

Hence even though the <u>average</u> waiting time is 5 minutes, only 37% of shoppers wait 5 minutes or more; in fact, the <u>median</u> waiting time is only 3.5 minutes (hw7), i.e. half the shoppers wait less than 3.5 minutes and half wait more.



The average waiting time ($\mu = 5$) is greater than the median waiting time ($m = 3.5$) because some of the shoppers who wait more than 3.5 minutes actually wait a lot longer (e.g. 10 minutes).

2 differential equations

some famous differential equations

Newton's 2nd law : particle moving in a force field

Maxwell's equations : electromagnetic waves

Schrödinger equation : quantum mechanics

2.1 1st order equations

Consider a 1st order differential equation of the form $y' = f(y)$, where $y = y(t)$ is a function of time t, and $y' = y'(t) = \frac{dy}{dt}(t)$ is the 1st derivative.

examples

$y' = y$

$y' = -y$

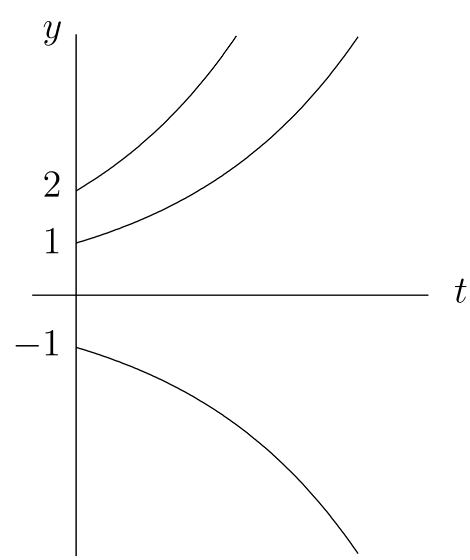
$y' = 1 - y^2$

1. The function $y(t)$ represents a quantity that varies in time (e.g. population, temperature, value of an investment), and the differential equation $y' = f(y)$ relates the rate of change of $y(t)$ to the value of $y(t)$.

2. A function $y(t)$ is a solution of the differential equation if $y'(t) = f(y(t))$ for all t; we also say that $y(t)$ satisfies the differential equation.

example : Is $y(t)$ a solution of the equation $y' = y$? 25 Fri 10/11

$y(t)$	yes/no?	initial condition
e^t	yes	$y(0) = 1$
$e^t + 1$	no	
$2e^t$	yes	$y(0) = 2$
e^{2t}	no	
$-e^t$	yes	$y(0) = -1$
e^{-t}	no	
ce^t	yes	$y(0) = c$



A differential equation $y' = f(y)$ has infinitely many solutions $y(t)$, but for any constant c, there is a unique solution with initial condition $y(0) = c$.

example

The function $y(t) = \tanh t$ satisfies the equation $y' = 1 - y^2$ with $y(0) = 0$.

proof : $y' = \dfrac{d}{dt}\tanh t = \dfrac{d}{dt}\dfrac{\sinh t}{\cosh t} = \dfrac{\cosh^2 t - \sinh^2 t}{\cosh^2 t} = 1 - \tanh^2 t = 1 - y^2$ ok

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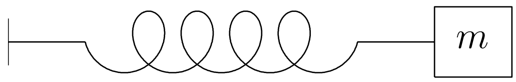
We can also consider 2nd order differential equations, i.e. equations involving y''.

example : spring-mass system



$y(t)$: displacement of spring from its natural length

$$\left.\begin{array}{l} \text{Newton's 2nd law : } F = m\,y'' \\ \text{Hooke's law : } F = -ky \ (k > 0\,, \text{restoring force}) \end{array}\right\} \Rightarrow \ m\,y'' = -ky$$

look for $y(t) = \cos\omega t \ \Rightarrow \ m \cdot -\omega^2 \cancel{\cos\omega t} = -k\cancel{\cos\omega t} \ \Rightarrow \ \omega = \sqrt{k/m}$

The system oscillates with frequency ω; this is simple harmonic motion.

other examples : pendulum, Earth's orbit, heartbeat, ... (Math 216/286/316)

2.2 exponential growth/decay

Let $y(t)$ be a population size at time t and assume the population changes at a rate proportional to its size, i.e. $y' = ky$, where k is a constant.

$k > 0 \ \Rightarrow \ y' > 0 \ \Rightarrow$ population grows

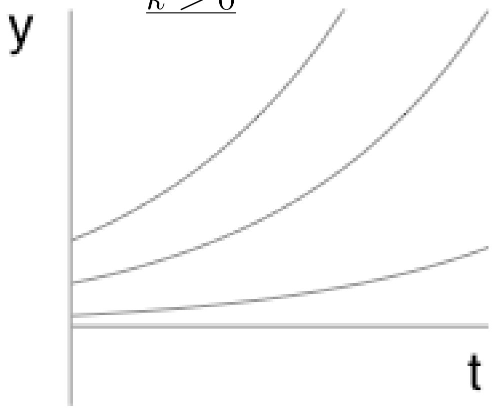
$k < 0 \ \Rightarrow \ y' < 0 \ \Rightarrow$ population decays

solution : $y' = ky \ \Rightarrow \ \dfrac{dy}{dt} = ky \ \Rightarrow \ \dfrac{dy}{y} = kdt$: separation of variables

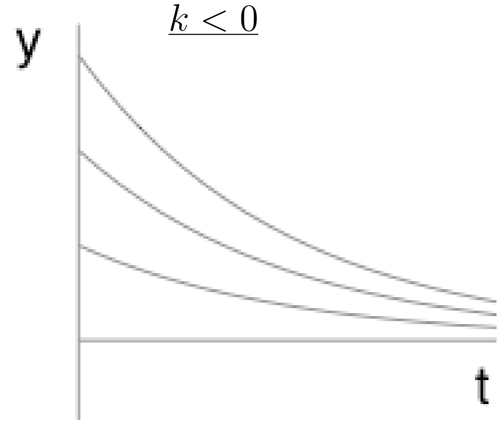
$$\Rightarrow \ \ln y = kt + c \ \Rightarrow \ y = e^{kt+c} = e^{kt} \cdot e^c = Ae^{kt}$$

$t = 0 \ \Rightarrow \ y(0) = A \ \Rightarrow \ y(t) = y(0)e^{kt}$, check ...

$k > 0$



$k < 0$



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<u>example</u>

A bacteria culture starts with 1000 cells and grows at a rate proportional to its size. If there are 2500 cells after 2 hours, how many cells are there after 4 hours?

$y(t)$: number of cells after t hours

$y(0) = 1000\ ,\ y(2) = 2500 \Rightarrow y(4) \neq 4000$

$y(t) = y(0)\,e^{kt} = 1000\,e^{kt}$

$y(2) = 1000\,e^{2k} = 2500 \Rightarrow e^{2k} = 2.5 \Rightarrow 2k = \ln 2.5 \Rightarrow k = \frac{1}{2}\ln 2.5 = 0.46$

$y(t) = 1000\,e^{0.46t} \Rightarrow y(4) = 1000e^{0.46\cdot 4} = 6296.54$ cells

alternative : $e^{kt} = e^{(\frac{1}{2}\ln 2.5)t} = (e^{\ln 2.5})^{t/2} = (2.5)^{t/2}$, recall : $e^{ab} = (e^a)^b$

$y(t) = 1000(2.5)^{t/2}$, check : $y(0) = 1000\,,\ y(2) = 2500$ <u>ok</u>

$y(4) = 1000(2.5)^2 = 1000(2+0.5)^2 = 1000(4+2+0.25) = 1000\cdot 6.25 = 6250$ cells

---

<u>example</u>

A radioactive sample has mass 100 mg and decays at a rate proportional to its size with half-life 1600 years (radium-226).

a) Find the mass remaining after t years.

$y(t)$: mass (mg) after t years

$y(t) = y(0)\,e^{kt} = 100\,e^{kt}$

$y(1600) = 100\,e^{1600\,k} = 50 \Rightarrow e^{1600\,k} = \frac{1}{2} \Rightarrow 1600\,k = \ln\frac{1}{2} = -\ln 2$

$k = \frac{-\ln 2}{1600} = -0.00043 \Rightarrow y(t) = 100\,e^{-0.00043\,t}$

alternative : $y(t) = 100\,e^{(-\ln 2/1600)t} = 100\,(e^{\ln 2})^{-t/1600} = 100\cdot 2^{-t/1600}$

check : $y(0) = 100\cdot 2^0 = 100\ ,\ y(1600) = 100\cdot 2^{-1} = 50$ <u>ok</u>

b) How much mass is there after 800 years? (half of a half-life)

$y(800) = 100\cdot 2^{-800/1600} = 100\cdot 2^{-1/2} = \frac{100}{\sqrt{2}} = 71$ mg

c) When will the sample be reduced to 25 mg?

$y(t) = 100\cdot 2^{-t/1600} = 25 \Rightarrow 2^{-t/1600} = \frac{1}{4} = 2^{-2} \Rightarrow \frac{-t}{1600} = -2$

$\Rightarrow t = 3200$ years $= 2$ half-lives

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example : compound interest

x : initial investment , r : annual interest rate

compounded annually

after 1 year, the investment is worth $x + rx = x(1+r)$

. . . . . 2 . . . . $x(1+r)^2$

. . . . . t . . . . $x(1+r)^t$

compounded semi-annually : 2 times/year

after $\frac{1}{2}$ year, the investment is worth $x + \frac{r}{2}x = x(1+\frac{r}{2})$

. . . . . 1 . . . . $x(1+\frac{r}{2})^2$

. . . . . t . . . . $x(1+\frac{r}{2})^{2t}$

note : $(1+\frac{r}{2})^2 = 1 + r + \frac{r^2}{4} > 1 + r$: more frequent compounding is good

compounded daily : 365 times/year

after $\frac{1}{365}$ year, the investment is worth $x + \frac{r}{365}x = x(1+\frac{r}{365})$

. . . . . $\frac{2}{365}$ . . . . $x(1+\frac{r}{365})^2$

. . . . . 1 . . . . $x(1+\frac{r}{365})^{365}$

. . . . . t . . . . $x(1+\frac{r}{365})^{365t}$

compounded continuously

after t years, the investment is worth

$$\lim_{n\to\infty} x(1+\tfrac{r}{n})^{nt} = x\Big(\lim_{n\to\infty}(1+\tfrac{r}{n})^n\Big)^t \overset{\text{hw6}}{=} x(e^r)^t = xe^{rt}$$

example : $x = \$1000$, $r = 0.1$ (i.e. 10%) , $t = 10$ years

compounded annually : $x(1+r)^t = 1000 \cdot (1.1)^{10} = \$2593.74$

compounded continuously : $xe^{rt} = 1000 \cdot e^{0.1 \cdot 10} = \$2718.28$

1. $e^{0.1} = 1.1052 = 1 + 0.1052 \Rightarrow$ the equivalent annual interest rate is 10.52%

$\Rightarrow$ 10% compounded continuously is the same as 10.52% compounded annually

2. continuous compounding : $xe^{rt} \Leftrightarrow y(t) = y(0)e^{rt} \Leftrightarrow y' = ry$

definition

A differential equation $y' = f(y)$ has a constant solution $y(t) = c \Leftrightarrow f(c) = 0$; it is also called an equilibrium solution.

example

$y' = 1 - y^2$ has 2 constant solutions : $c = \pm 1$

definition

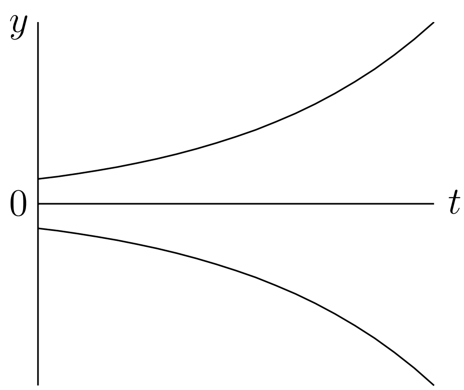
A constant solution $y(t) = c$ is stable $\Leftrightarrow \lim_{t\to\infty} y(t) = c$ for all nearby solutions; otherwise, it is unstable.

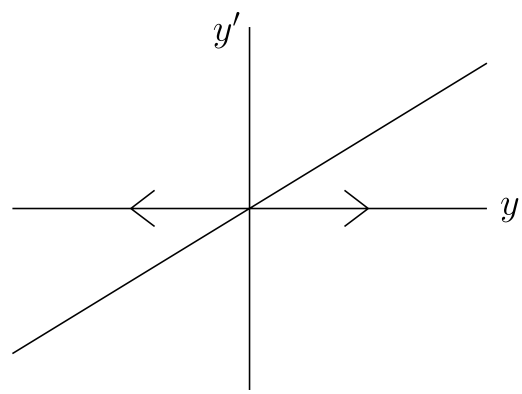
example

$y' = ky \Rightarrow$ the only constant solution is $c = 0$; is it stable or unstable?

recall : $y(t) = y(0)e^{kt}$ 28 Mon 10/21

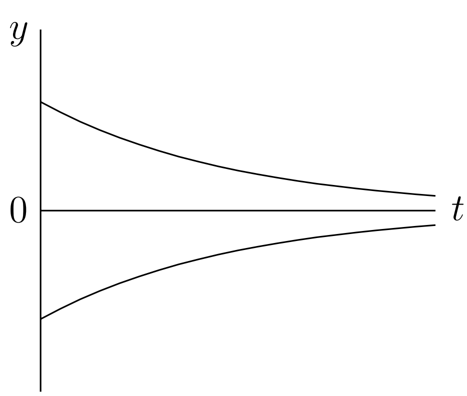
case 1 : $k > 0 \Rightarrow 0$ is an unstable constant solution

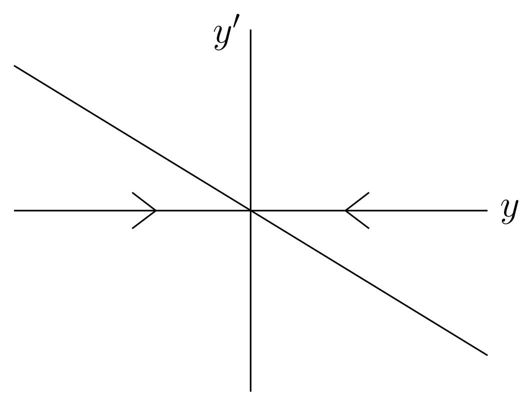




The yy'-plane is called the phase plane; it describes the dynamics of $y(t)$ without the need to know the explicit formula for $y(t)$; the arrows point in the direction of increasing time.

case 2 : $k < 0 \Rightarrow 0$ is a stable constant solution





2.3 Newton's law of cooling/heating

The rate of change of temperature of an object is proportional to the temperature difference between the object and its surroundings.

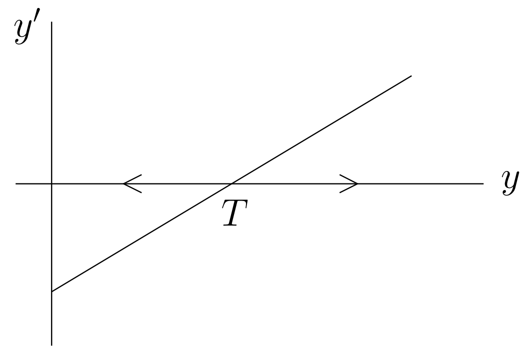
$y(t)$: temperature of object

T : temperature of surroundings

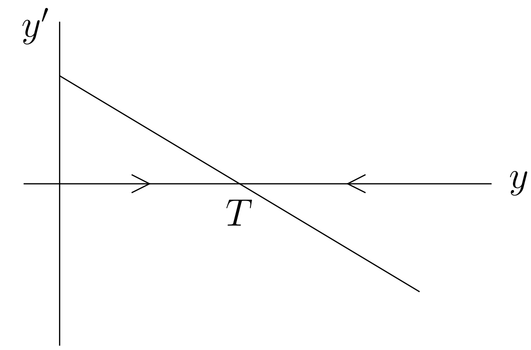
$\Rightarrow\ y' = k(y - T)$

There is 1 constant solution given by $y(t) = T$. Is it stable or unstable?

<u>case 1</u> : $k > 0 \Rightarrow T$ is unstable



<u>case 2</u> : $k < 0 \Rightarrow T$ is stable



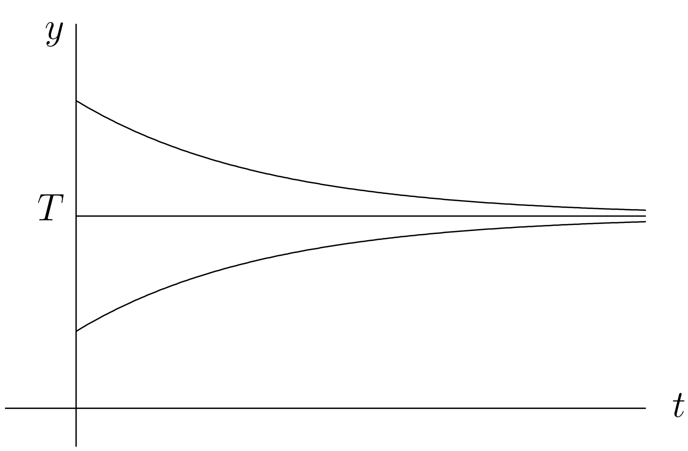
The case $k > 0$ is unphysical, so in this context we assume $k < 0$.

<u>solution</u>

$$\frac{dy}{dt} = k(y - T) \Rightarrow \frac{dy}{y - T} = k\,dt \Rightarrow \ln(y - T) = kt + c \Rightarrow y - T = e^{kt + c} = Ae^{kt}$$

$t = 0 \Rightarrow y_0 - T = A$, where $y_0 = y(0)$

$\Rightarrow\ y - T = (y_0 - T)e^{kt} \Rightarrow y(t) = T + (y_0 - T)e^{kt}$, check ...



$y_0 > T \Rightarrow$ object is cooling

$y_0 < T \Rightarrow$ object is heating

This confirms the phase plane analysis.

<u>example</u>: It takes 30 minutes for a can of soda (pop) to cool from 30°C to 24°C in a refrigerator at 10°C. Find the soda temperature 60 minutes after it starts to cool.

$y(t)$: soda temperature t minutes after it starts to cool

$y(t) = T + (y_0 - T)e^{kt} = 10 + (30 - 10)e^{kt} = 10 + 20e^{kt}$ 29 Tues 10/22

$y(30) = 10 + 20e^{30k} = 24 \Rightarrow 20e^{30k} = 14 \Rightarrow e^{30k} = 0.7 \Rightarrow 30k = \ln(0.7)$

$\Rightarrow k = \frac{1}{30}\ln(0.7) = -0.012$

$y(t) = 10 + 20e^{-0.012\,t} \Rightarrow y(60) = 10 + 20e^{-0.012\cdot 60} = 10 + 20e^{-0.72} = 19.74°\text{C}$

alternative : $y(t) = 10 + 20e^{\frac{1}{30}\ln(0.7)t} = 10 + 20(e^{\ln(0.7)})^{t/30} = 10 + 20\cdot(0.7)^{t/30}$

$y(60) = 10 + 20\cdot(0.7)^2 = 10 + 20\cdot 0.49 = 19.80°\text{C}$

In the first 30 minutes the soda cools 6°C, but in the second 30 minutes it cools only 4°C; this is because as $y(t) \to T$, $y'(t) \to 0$, i.e. the rate of cooling decreases.

<u>example</u>: A 5000 L tank contains brine with salt concentration 0.004 kg/L. Seawater with salt concentration 0.03 kg/L starts pouring into the tank at a rate of 25 L/min. The solution is well mixed, and it drains from the tank at the same rate that the seawater enters. Find the amount of salt in the tank after 30 minutes.

$y(t)$: amount of salt (kg) in tank after t minutes

$y(0) = 0.004\,\frac{\text{kg}}{\text{L}}\cdot 5000\text{ L} = 20\text{ kg}$

y' = rate of change of amount of salt in tank $(\frac{\text{kg}}{\text{min}})$

= (rate coming in) - (rate going out)

rate coming in $= 0.03\,\frac{\text{kg}}{\text{L}}\cdot 25\,\frac{\text{L}}{\text{min}} = 0.75\,\frac{\text{kg}}{\text{min}}$

rate going out $= \frac{y\,\text{kg}}{5000\,\text{L}}\cdot 25\,\frac{\text{L}}{\text{min}} = 0.005\,y\,\frac{\text{kg}}{\text{min}}$

$y' = 0.75 - 0.005\,y = -0.005(y - 150) = k(y - T)$

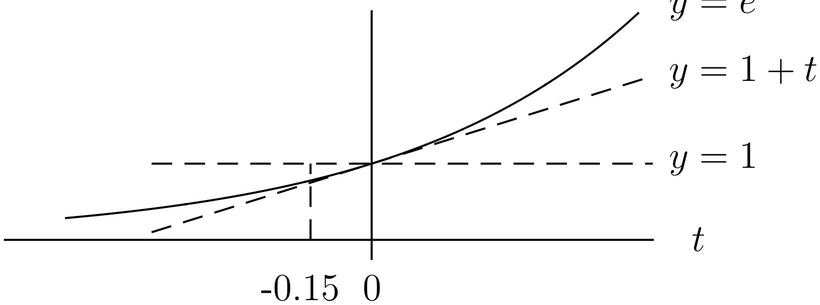
$y(t) = T + (y_0 - T)e^{kt} = 150 + (20 - 150)e^{-0.005t} = 150 - 130e^{-0.005t}$

$y(30) = 150 - 130e^{-0.005\cdot 30} = 150 - 130e^{-0.15} = 38\text{ kg}$

question : What does T represent in this problem?

answer : $T = \lim_{t\to\infty} y(t) =$ amount of salt in tank when completely filled with seawater $= 0.03\,\frac{\text{kg}}{\text{L}}\cdot 5000\text{ L} = 150\text{ kg}$

note : $e^{-0.15} = 0.8607$ - what is the calculator doing?



$e^{-0.15} \approx e^0 = 1$, but the tangent line gives a better approximation

$e^{-0.15} \approx 1 - 0.15 = 0.85$ (more later)

2.4 logistic equation

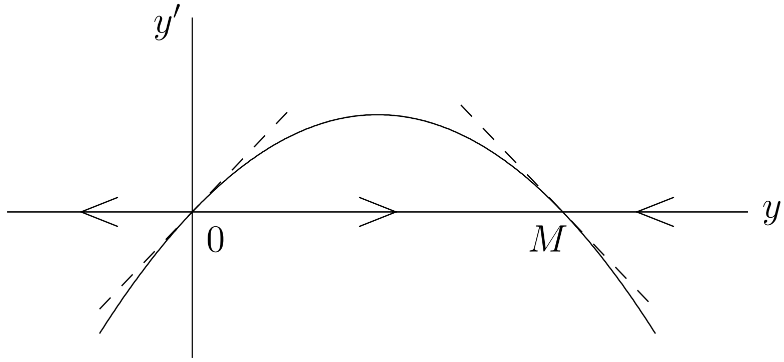
30 Wed 10/23

$y(t)$: population , $M > 0$: maximum population (due to finite resources)

$y' = k\Big(1 - \dfrac{y}{M}\Big)y$: <u>logistic equation</u>

assume $k > 0$, then $k\Big(1 - \dfrac{y}{M}\Big)$ is a variable growth rate

There are 2 constant solutions : $y = 0$, $y = M$; are they stable or unstable?

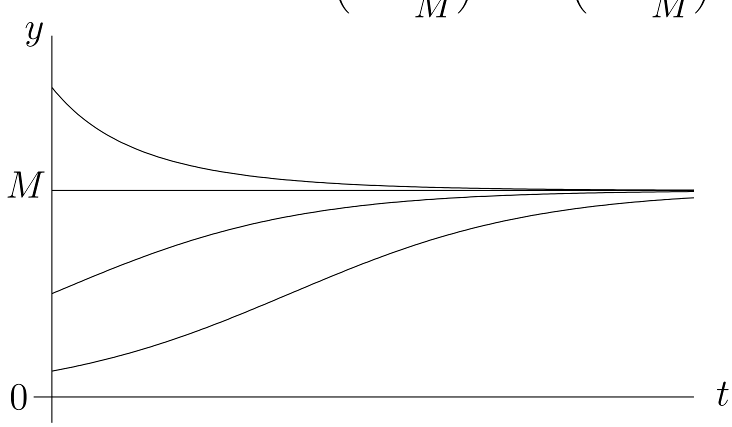


$\Rightarrow \begin{cases} 0 & \text{is unstable} \\ M & \text{is stable} \end{cases}$

<u>linear stability analysis</u>

if $y \approx 0$, then $y' = k\Big(1 - \dfrac{y}{M}\Big)y \approx ky$: exponential growth

if $y \approx M$, then $y' = k\Big(1 - \dfrac{y}{M}\Big)y \approx k\Big(1 - \dfrac{y}{M}\Big)M = -k(y - M)$: cooling/heating



<u>solution</u>

$$\frac{dy}{dt} = k\Big(1 - \frac{y}{M}\Big)y = k\Big(\frac{M-y}{M}\Big)y \;\Rightarrow\; \frac{M\,dy}{(M-y)y} = k\,dt$$

$$\frac{M}{(M-y)y} = \frac{a}{M-y} + \frac{b}{y} = \frac{ay + b(M-y)}{(M-y)y} = \frac{bM + y(a-b)}{(M-y)y}$$

$$\left.\begin{array}{l} bM = M \;\Rightarrow b = 1 \\ a - b = 0 \Rightarrow a = 1 \end{array}\right\} \Rightarrow \frac{M}{(M-y)y} = \frac{1}{M-y} + \frac{1}{y} \quad , \text{ check } \ldots$$

$$\int \frac{M\,dy}{(M-y)y} = \int \frac{dy}{M-y} + \int \frac{dy}{y} = -\ln(M-y) + \ln y = \ln\left(\frac{y}{M-y}\right) = kt + c$$

$$\Rightarrow \frac{y}{M-y} = e^{kt+c} = Ae^{kt}$$

$$t = 0 \;\Rightarrow\; \frac{y_0}{M-y_0} = A$$

$$\frac{y}{M-y} = \frac{y_0}{M-y_0}e^{kt} \;\Rightarrow\; y(M-y_0) = (M-y)y_0e^{kt}$$

$$\Rightarrow y\Big(M - y_0 + y_0e^{kt}\Big) = My_0e^{kt}$$

$$\Rightarrow y = \frac{My_0e^{kt}}{M - y_0 + y_0e^{kt}}$$

$$\Rightarrow y(t) = \frac{My_0}{(M-y_0)e^{-kt} + y_0} \quad , \text{ check : } y(0) = y_0 \;, \;\ldots$$

$$\left.\begin{array}{l} \text{1. if } y_0 = 0, \text{ then } y(t) = 0 \\ \quad\text{if } y_0 = M, \text{ then } y(t) = M \end{array}\right\} \text{these are the constant solutions}$$

$$\text{2. if } 0 < y_0 < M, \text{ then } \lim_{t\to\infty} y(t) = M \;\Rightarrow \left\{\begin{array}{l} 0 \;\text{ is unstable} \\ M \text{ is stable} \end{array}\right.$$

This confirms the phase plane and linear stability analysis.

2.5 Euler's method

31 Fri 10/25

This is a step-by-step numerical method for solving differential equations.

applications : spacecraft trajectory planning, weather prediction, ...

consider $y' = f(y)$, $y(0) = y_0$, goal : approximate $y(t)$

choose Δt : time step , set $t_n = n\Delta t$, $n = 0, 1, 2, \ldots$

define $y_n = y(t_n)$: exact solution at time t_n

u_n : numerical solution at time t_n (approximation to y_n)

$\dfrac{u_{n+1} - u_n}{\Delta t} = f(u_n)$: Euler's method

$\Rightarrow u_{n+1} = u_n + \Delta t\, f(u_n)$, $u_0 = y_0$, this determines $u_1, u_2, \ldots$

<u>example</u>

$y' = y$, $y_0 = 1 \Rightarrow y(t) = e^t$, $f(y) = y$

$u_{n+1} = u_n + \Delta t\, u_n = (1 + \Delta t)u_n$

$u_0 = 1$

$u_1 = (1 + \Delta t)u_0 = 1 + \Delta t$

$u_2 = (1 + \Delta t)u_1 = (1 + \Delta t)^2$

$\ldots$

$u_n = (1 + \Delta t)^n$

for example, suppose $t_n = n\Delta t = 1$, then $y_n = y(t_n) = y(1) = e = 2.7183$

n	Δt	u_n	$\lvert y_n - u_n\rvert$	$\lvert y_n - u_n\rvert/\Delta t$
1	1	2.0000	0.7183	0.7183
2	1/2	2.2500	0.4683	0.9366
4	1/4	2.4414	0.2769	1.1075
8	1/8	2.5658	0.1525	1.2200
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
∞	0	e	0	$\frac{0}{0} = 1.3591 = \frac{e}{2}$, pf : exam 2 review sheet

1. $\displaystyle\lim_{\Delta t\to 0} u_n = \lim_{\Delta t\to 0}(1 + \Delta t)^n = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e \Rightarrow$ Euler's method <u>converges</u>

2. If the time step Δt decreases by a factor of $\frac{1}{2}$, then the error decreases by a factor of approximately $\frac{1}{2}$; hence the error is proportional to Δt and we say "the error $= O(\Delta t)$".

3. There are higher order methods, e.g. methods for which the error $= O(\Delta t^2)$. (Math 371/471, numerical methods)

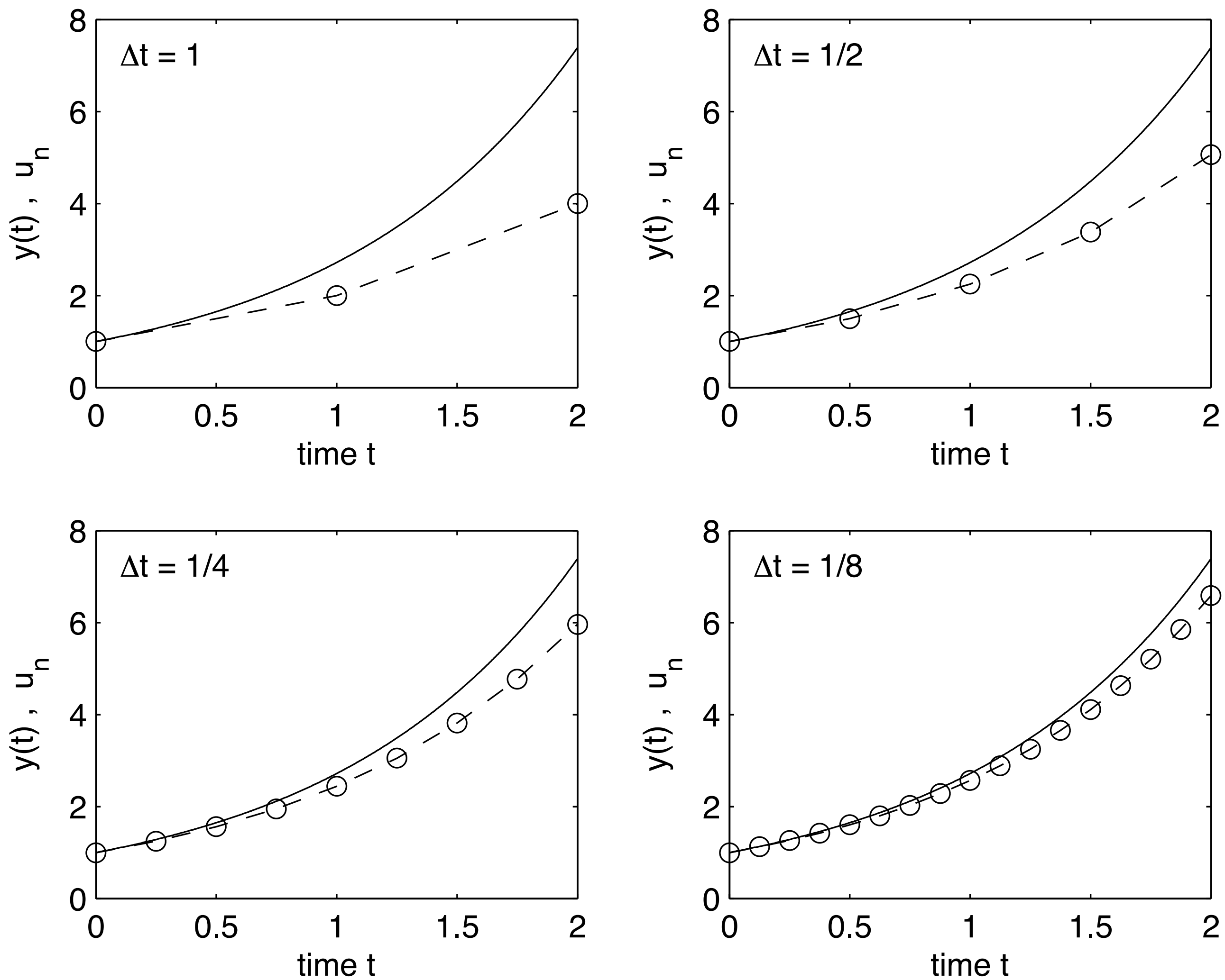
Euler's method

differential equation : $y' = y$, $y(0) = 1 \Rightarrow y(t) = e^t$

numerical method : $u_{n+1} = u_n + \Delta t \, u_n$, $u_0 = 1 \Rightarrow u_n = (1 + \Delta t)^n$

The solid line is the exact solution $y(t)$.

The circles are the numerical solution u_n.



1. For a fixed time t, the error decreases as the time step $\Delta t \to 0$.

2. For a fixed time step Δt, the error increases as time $t \to \infty$.

3 series

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10/28

preview

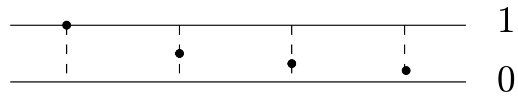
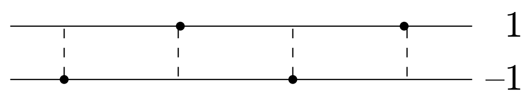
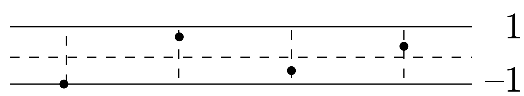
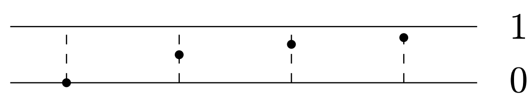
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ : \ \underline{\text{Taylor series}}$$

questions : meaning? utility?

3.1 sequences

def : A sequence is a list of numbers $\{a_1, a_2, a_3, \ldots\} = \{a_n\}$. If the sequence converges to a limit L, then we write $\lim_{n\to\infty} a_n = L$, or $a_n \to L$ as $n \to \infty$, and if the sequence does not converge, then we say it diverges.

example

	terms		converges?
$a_n = \frac{1}{n}$	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$		$L = 0$
$b_n = (-1)^n$	$-1, 1, -1, 1, \ldots$		diverges
$c_n = \frac{(-1)^n}{n}$	$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots$		$L = 0$
$d_n = 1 - \frac{1}{n}$	$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$		$L = 1$

comparison test

If $a_n \le b_n \le c_n$ for all n, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

proof : omit

example : $\lim_{n\to\infty} \frac{n!}{n^n} = \frac{\infty}{\infty} = 0$

proof :

n	1	2	3	$\cdots$
$\frac{n!}{n^n}$	$\frac{1}{1} = 1$	$\frac{2\cdot 1}{2\cdot 2} = \frac{1}{2}$	$\frac{3\cdot 2\cdot 1}{3\cdot 3\cdot 3} = \frac{2}{9}$	$\cdots$

$\Rightarrow 0 \le \frac{n!}{n^n} \le \frac{1}{n} \ \ldots$ ok

3.2 series

$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$: series

The starting index can be $n = 0$ or $n = 1$ or …

$\{s_n\}$: sequence of partial sums

$s_0 = a_0$

$s_1 = a_0 + a_1$

$s_2 = a_0 + a_1 + a_2$

$\vdots$

$s_n = a_0 + a_1 + a_2 + \cdots + a_n$

$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n$: If the limit exists, then the series converges; otherwise, the series diverges.

example : $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 2$

$s_0 = 1$

$s_1 = 1 + \frac{1}{2} = \frac{3}{2}$

$s_2 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$

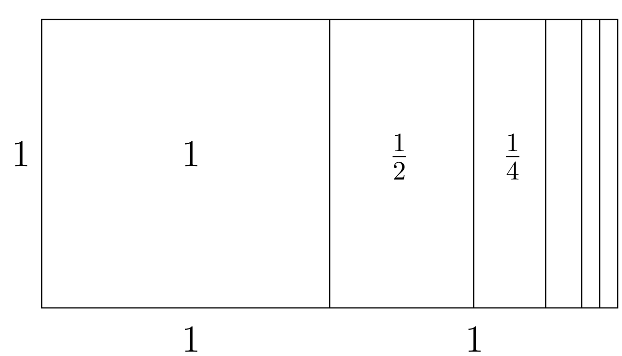
$s_3 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$

$\vdots$

$\Rightarrow \lim_{n\to\infty} s_n = 2$

proof

1. geometric proof :



$\Rightarrow$ area = 2

2. analytic proof : soon

<u>geometric series</u>

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots = \begin{cases} \dfrac{1}{1-r} & \text{if } -1 < r < 1 \\ \text{diverges if } r \geq 1 \text{ or } r \leq -1 \end{cases}$$

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<u>proof</u> : finite geometric series

recall : $\displaystyle\sum_{i=0}^{n} r^i = 1 + r + r^2 + \cdots + r^n = \begin{cases} \dfrac{1-r^{n+1}}{1-r} & \text{if } r \neq 1 \\ n+1 & \text{if } r = 1 \end{cases}$

what happens as $n \to \infty$?

1. if $-1 < r < 1$, then $\lim_{n\to\infty} r^{n+1} = 0$
2. if $r = 1$, then $\lim_{n\to\infty} (n+1)$ diverges
3. if $r > 1$ or $r \leq -1$, then $\lim_{n\to\infty} r^{n+1}$ diverges <u>ok</u>

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = 2 \text{ (as before)}$$

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$$

$$2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots = \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^n} = 2 \cdot \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 2 \cdot \frac{1}{1-\frac{2}{3}} = 6$$

<u>example</u>

A rubber ball is dropped from height 1, then bounces back to height h, where $0 < h < 1$, and continues bouncing after that. Find the total distance the ball travels.

initial height $= 1$

height after 1 bounce $= h$

$\cdots\cdots\cdots\cdots$ 2 bounces $= h^2$

$\cdots\cdots\cdots\cdots$ n bounces $= h^n$

$$\text{distance} = 1 + 2h + 2h^2 + \cdots = 2(1 + h + h^2 + \cdots) - 1 = 2 \cdot \frac{1}{1-h} - 1$$

$$= \frac{2-(1-h)}{1-h} = \frac{1+h}{1-h}$$

for example : $h = \frac{1}{2} \Rightarrow \text{distance} = \frac{3/2}{1/2} = 3$

$h \to 0 \Rightarrow \cdots\cdots\cdots \to 1$

$h \to 1 \Rightarrow \cdots\cdots\cdots \to \infty$

<u>harmonic series</u>

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots : \text{ diverges}$$

<u>proof</u>

$s_1 = 1$

$s_2 = s_1 + \frac{1}{2} = \frac{3}{2}$

$s_4 = s_2 + \frac{1}{3} + \frac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = 2$

$s_8 = s_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 2 + 4 \cdot \frac{1}{8} = \frac{5}{2}$

$s_{16} = s_8 + \frac{1}{9} + \cdots + \frac{1}{16} > \frac{5}{2} + 8 \cdot \frac{1}{16} = 3$

$\dots$

$\Rightarrow \lim_{n\to\infty} s_n = \infty$ <u>ok</u>

<u>theorem</u> : If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

<u>proof</u>

set $\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n = s$

$s_n = a_0 + a_1 + \cdots + a_n$

$s_{n-1} = a_0 + a_1 + \cdots + a_{n-1}$

$s_n - s_{n-1} = a_n$

$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$ <u>ok</u>

1. The converse is false, i.e. if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=0}^{\infty} a_n$ may converge or diverge. 34 Wed 10/30

$$\left.\begin{array}{l} \sum_{n=0}^{\infty} \frac{1}{2^n} : \text{ converges} \\ \sum_{n=1}^{\infty} \frac{1}{n} : \text{ diverges} \end{array}\right\} \text{both satisfy } \lim_{n\to\infty} a_n = 0$$

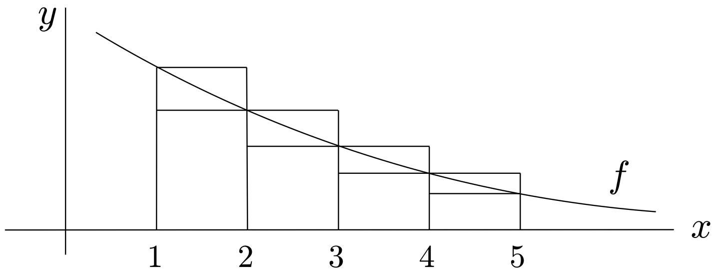
2. However, if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

3.3 convergence tests for series

integral test : Let $a_n = f(n)$, where $f(x)$ is positive and decreasing for $x \geq 1$.

Then $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \int_1^\infty f(x)dx$ converges.

proof



$$a_2 + a_3 + a_4 + a_5 < \int_1^5 f(x)dx \; < \; a_1 + a_2 + a_3 + a_4 \quad \ldots \quad \underline{\text{ok}}$$

---

example : $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$: converges

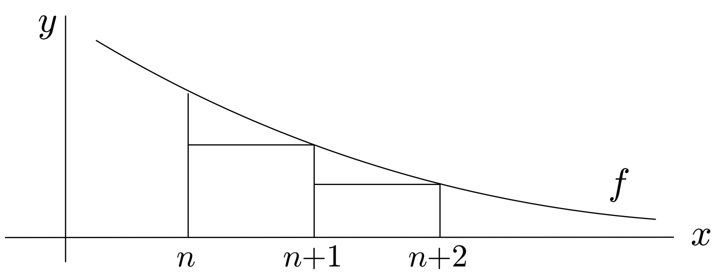
proof : $\frac{1}{n^2} = f(n) \Rightarrow f(x) = \frac{1}{x^2} \Rightarrow \int_1^\infty f(x)dx = \int_1^\infty \frac{dx}{x^2}$: converges , $p = 2$ ok

note : We don't know the value of the sum s, but we can approximate it by s_n.

for example : $s_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = \frac{144 + 36 + 16 + 9}{144} = \frac{205}{144} = 1.42361111$

question 1 : How large is the error?

$$s - s_n \; = \; \cancel{a_1} + \cancel{\cdots} + \cancel{a_n} + a_{n+1} + \cdots - (\cancel{a_1} + \cancel{\cdots} + \cancel{a_n}) \; = \; a_{n+1} + \cdots < \int_n^\infty f(x)dx$$



$$\Rightarrow \; 0 < s - s_n < \int_n^\infty f(x)dx \; : \; \underline{\text{error bound}} \; \Rightarrow \; s_n < s < s_n + \int_n^\infty f(x)dx$$

$$n = 4 \; \Rightarrow \; \int_4^\infty \frac{dx}{x^2} = -\frac{1}{x}\bigg|_4^\infty = \frac{1}{4} = 0.25 \; \Rightarrow \; 1.42361111 < s < 1.67361111$$

question 2 : How large must n be to ensure the error is less than 10^{-3}?

$$s - s_n < \int_n^\infty \frac{dx}{x^2} = -\frac{1}{x}\bigg|_n^\infty = \frac{1}{n} = 10^{-3} \; \Rightarrow \; n = 1000$$

$$\Rightarrow \; s_{1000} < s < s_{1000} + 10^{-3} \; \Rightarrow \; 1.64393457 < s < 1.64493457$$

in fact, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.64493407 \begin{cases} \underline{\text{Basel problem}} \\ \text{solved by Leonhard Euler in 1735} \end{cases}$

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p-test for series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \le 1 \end{cases}$$

proof : integral test , $f(x) = \dfrac{1}{x^p}$, $\displaystyle\int_1^\infty \frac{dx}{x^p} \cdots$ ok

---

comparison test

Assume $0 \le a_n \le b_n$ for all n.

1. If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.
2. If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

proof : $\displaystyle\sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n \quad \ldots$ ok

example : $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots$: converges

proof : $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$: converges , $p = 2$ ok

in fact $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$: hw10

3.4 alternating series

definition: $\displaystyle\sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \cdots$, where $a_n > 0$

alternating series test

If a_n satisfies :

1. $a_n > 0$,
2. $a_{n+1} < a_n$, i.e. $\{a_n\}$ is a decreasing sequence,
3. $\displaystyle\lim_{n\to\infty} a_n = 0$,

then $\displaystyle\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

<u>alternating harmonic series</u>

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \ : \text{ converges}$$

<u>proof</u>

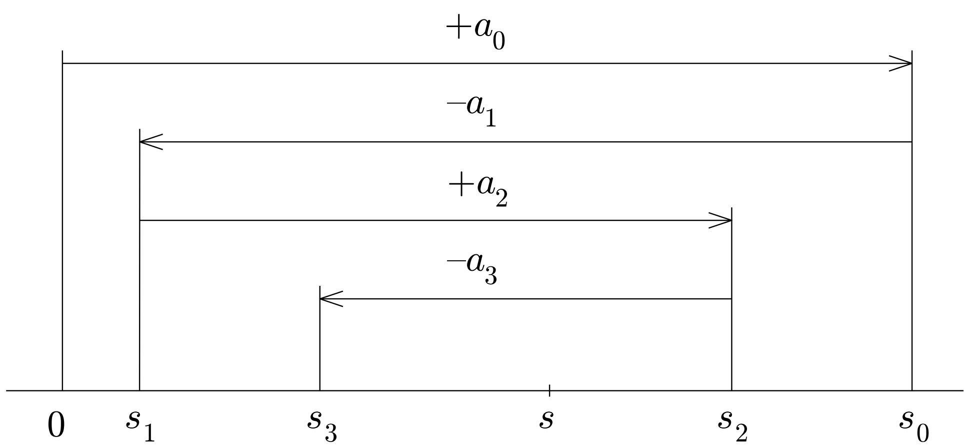
$$a_n = \frac{1}{n+1}$$

$$\left.\begin{array}{l} 1.\ \dfrac{1}{n+1} > 0 \\[2ex] 2.\ \dfrac{1}{n+2} < \dfrac{1}{n+1} \\[2ex] 3.\ \displaystyle\lim_{n\to\infty} \frac{1}{n+1} = 0 \end{array}\right\} \Rightarrow \text{ series converges by AST}$$

$$\text{recall} \ : \ \sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \ : \ \text{diverges}$$

<u>proof</u> (AST)

$$\sum_{n=0}^{\infty} (-1)^n a_n = \lim_{n\to\infty} s_n = s \ : \ \text{we need to show the limit exists}$$



$s_0 = a_0$

$s_1 = a_0 - a_1$

$s_2 = a_0 - a_1 + a_2$

$s_3 = a_0 - a_1 + a_2 - a_3$

$\ldots$

It follows that $\lim_{n\to\infty} s_n = s$ exists. <u>ok</u>

<u>error bound</u> for alternating series

Let $\sum\limits_{n=0}^{\infty}(-1)^n a_n = s$, where a_n satisfies the conditions of the AST.

Then $|s-s_n| < a_{n+1}$, i.e. the error is less than the first neglected term. 36 Mon 11/4

note : $|x|<1 \Leftrightarrow -1<x<1$

$$|s-s_n| < a_{n+1} \Leftrightarrow -a_{n+1} < s-s_n < a_{n+1} \Leftrightarrow s_n - a_{n+1} < s < s_n + a_{n+1}$$

<u>example</u> : $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{n+1} = 1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\cdots = s$: converges by AST

$$s_3 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4} = \frac{12-6+4-3}{12} = \frac{7}{12} = 0.5833$$

error bound $\Rightarrow |s-s_3| < a_4 = \frac{1}{5} = 0.2 \Rightarrow 0.3833 < s < 0.7833$

in fact, $s = \ln 2 = 0.6931$ (proof : soon)

<u>proof</u> (error bound)

$$s = a_0 - a_1 + a_2 - \cdots + (-1)^n a_n + (-1)^{n+1} a_{n+1} + \cdots$$

$$s_n = a_0 - a_1 + a_2 - \cdots + (-1)^n a_n$$

$$\begin{aligned} s - s_n &= (-1)^{n+1}a_{n+1} + (-1)^{n+2}a_{n+2} + \cdots \\ &= (-1)^{n+1}(a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots) \end{aligned}$$

$$\begin{aligned} |s-s_n| &= |(a_{n+1}-a_{n+2}) + (a_{n+3}-a_{n+4}) + \cdots| \\ &= (a_{n+1}-a_{n+2}) + (a_{n+3}-a_{n+4}) + \cdots \\ &= a_{n+1} - (a_{n+2}-a_{n+3}) - (a_{n+4}-a_{n+5}) - \cdots \\ &< a_{n+1} \quad \underline{\text{ok}} \end{aligned}$$

This error bound is different than the one we derived for series of the form $\sum\limits_{n=1}^{\infty} a_n$, where $a_n = f(n)$ and $f(x)$ is positive and decreasing; in that case we saw that $0 < s - s_n < \int_n^{\infty} f(x)dx \Leftrightarrow s_n < s < s_n + \int_n^{\infty} f(x)dx$.

3.5 ratio test : suppose $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = L$

1. If $L<1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.
2. If $L>1$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.
3. If $L=1$, the ratio test is inconclusive; the series may converge or diverge.

note : the terms a_n can be positive or negative examples

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \ : \ \text{converges} \ , \ \text{geometric series} \ , \ r = \frac{1}{2}$$

$$a_n = \frac{1}{2^n} \ \Rightarrow \ \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} \right| = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

ratio test $\Rightarrow$ series converges

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \ : \ \text{diverges} \ , \ \text{harmonic series}$$

$$a_n = \frac{1}{n} \ \Rightarrow \ \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{n+1} \cdot n = 1$$

ratio test is inconclusive

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \ : \ \text{converges} \ , \ p\text{-test} \ , \ p = 2$$ 37 Tues 11/5

$$a_n = \frac{1}{n^2} \ \Rightarrow \ \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{(n+1)^2} \cdot n^2 = 1$$

ratio test is inconclusive

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots \ : \ \text{converges}$$

$$a_n = \frac{1}{n!} \ \Rightarrow \ \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot n! = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

ratio test $\Rightarrow$ series converges

---

proof (ratio test, just the idea)

assume $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then $\left| \frac{a_{n+1}}{a_n} \right| \sim L$ as $n \to \infty$

$\Rightarrow \ |a_{n+1}| \sim L|a_n|$

$\Rightarrow \ |a_{n+2}| \sim L|a_{n+1}| \sim L^2|a_n|$

$\Rightarrow \ |a_{n+3}| \sim L|a_{n+2}| \sim L^3|a_n|$

$\cdots$

$$\begin{aligned} \Rightarrow \ & |a_n| + |a_{n+1}| + |a_{n+2}| + |a_{n+3}| + \cdots \\ & \sim |a_n| + L|a_n| + L^2|a_n| + L^3|a_n| + \cdots \\ & = |a_n|(1 + L + L^2 + L^3 + \cdots) \quad \ldots \ : \ \text{converges if } L < 1 \end{aligned}$$

If $L > 1$, then $\lim_{n\to\infty} |a_n| = \infty$, so the series diverges.

If $L = 1 \ldots$ inconclusive. ok

3.6 power series

A power series has the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$.

c_n : coefficients , x : variable

For a given value of x, the power series may converge or diverge.

---

example 1

$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$: converges for $-1 < x < 1$

geometric series , $r = x$

---

example 2

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$: converges for $-\infty < x < \infty$

$a_n = \frac{x^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 \text{ for all } x$$

---

example 3 (more later)

$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = 1 - \frac{x^2}{4\cdot 1} + \frac{x^4}{16\cdot 4} - \frac{x^6}{64\cdot 36} + \cdots$: converges for $-\infty < x < \infty$

$a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| = \lim_{n\to\infty} \frac{|x|^2}{4(n+1)^2} = 0 \text{ for all } x$$

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A power series <u>centered at $x = a$</u> has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

$\sum_{n=0}^{\infty} (x-1)^n = 1 + (x-1) + (x-1)^2 + \cdots$: converges for $0 < x < 2$

geometric series , $r = x - 1$, $-1 < r < 1 \Leftrightarrow -1 < x - 1 < 1 \Leftrightarrow 0 < x < 2$

$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n} = (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots$: converges for $0 \le x < 2$

$$a_n = \frac{(x-1)^n}{n}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n}\right| = \lim_{n\to\infty} |x-1| \cdot \frac{n}{n+1} = |x-1| < 1$$

For $x = 0$ or $x = 2$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, so the ratio test is inconclusive.

$x = 0 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$: converges by AST

$x = 2 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$: diverges , $p = 1$

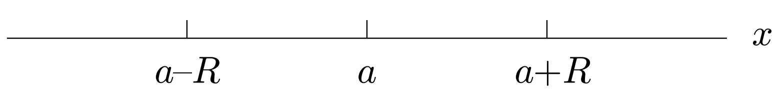
---

question : For what values of x does the power series converge?

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a| \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \frac{|x-a|}{R} < 1$$

$\Leftrightarrow |x-a| < R \Leftrightarrow -R < x - a < R \Leftrightarrow a - R < x < a + R$



R : <u>radius of convergence</u> , three possibilities

1. if $0 < R < \infty$, then the series $\begin{cases} \text{converges for } |x-a| < R \\ \text{diverges for } |x-a| > R \\ \text{may converge or diverge for } |x-a| = R \end{cases}$

2. if $R = \infty$, then the series converges for all x

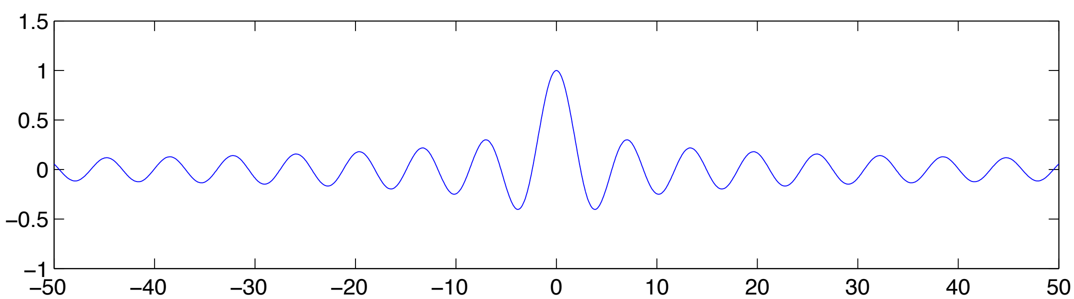
3. if $R = 0$, then the series converges for $x = a$ and diverges for $x \neq a$

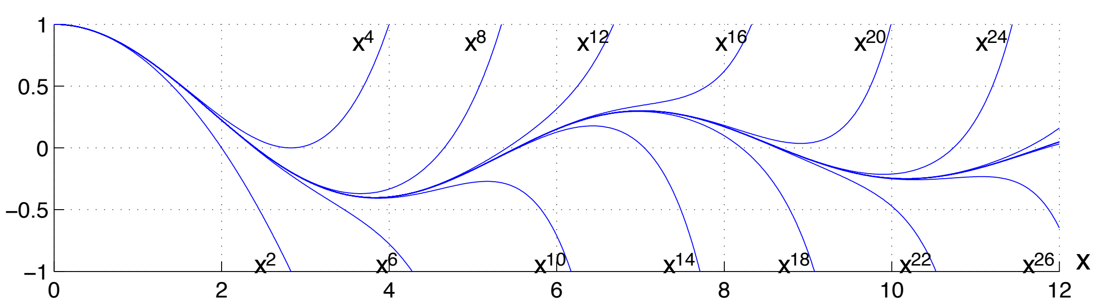
<u>interval of convergence</u> : set of all x for which the series converges

series	center	roc	ioc	sum
$\sum_{n=0}^{\infty} x^n$	$a=0$	$R=1$	$-1<x<1$	$\frac{1}{1-x}$
$\sum_{n=0}^{\infty} (x-1)^n$	$a=1$	$R=1$	$0<x<2$	$\frac{1}{1-(x-1)} = \frac{1}{2-x}$
$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$	$a=1$	$R=1$	$0\le x<2$	$-\ln(2-x)$: soon
$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$a=0$	$R=\infty$	$-\infty<x<\infty$	e^x : soon
$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$a=0$	$R=\infty$	$-\infty<x<\infty$	$J_0(x)$

Bessel function of order zero

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = 1 - \frac{x^2}{4\cdot 1} + \frac{x^4}{16\cdot 4} - \frac{x^6}{64\cdot 36} + \cdots$$





ripples on a lake, vibrations of a drum , Math 454 (partial differential equations)

<u>power series representation of a function</u>

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$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

problem : given $f(x)$, find c_n

<u>example</u>

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \text{ for } -1 < x < 1$$

Power series can be added, subtracted, multiplied, and divided, and the result is another power series.

<u>example</u>

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)^2 = \left(\sum_{n=0}^{\infty} x^n\right)^2 = (1 + x + x^2 + x^3 + \cdots)^2$$

$$= (1 + x + x^2 + x^3 + \cdots) \cdot (1 + x + x^2 + x^3 + \cdots)$$

$$= 1 + x(1+1) + x^2(1+1+1) + x^3(1+1+1+1) + \cdots$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n \text{ for } -1 < x < 1$$

Power series can also be differentiated and integrated.

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\int f(x)dx = c_0 x + \frac{c_1 x^2}{2} + \frac{c_2 x^3}{3} + \cdots = \sum_{n=0}^{\infty} \frac{c_n x^{n+1}}{n+1}$$

<u>theorem</u>

1. The power series for $f'(x)$ and $\int f(x)dx$ have the same roc as the power series for $f(x)$, but the endpoints of the ioc must be checked in each case.

2. Similar results hold for power series centered at $x = a$.

<u>proof</u> : ratio test ...

example

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \text{ for } -1 < x < 1$$

differentiate

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n \text{ for } -1 < x < 1$$

This agrees with the previous result obtained by multiplication.

integrate

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n} \text{ for } -1 \le x < 1$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots = -\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ for } -1 \le x < 1$$

$$x = 1 \Rightarrow \ln 0 = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots \ : \text{ diverges}$$

$$x = -1 \Rightarrow \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \ : \text{ converges}$$

$$x = \frac{1}{2} \Rightarrow \ln\frac{1}{2} = -\frac{1}{2} - \frac{1}{8} - \frac{1}{24} - \frac{1}{64} - \cdots = -\ln 2$$

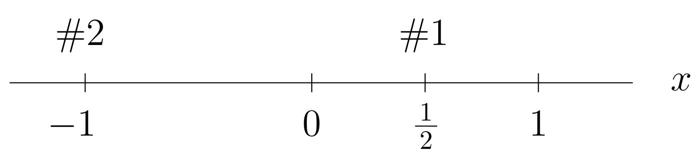
$$\ln 2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \cdots = 0.6931 \ : \ \#1$$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = 0.6931 \ : \ \#2$$

question : Which series converges faster?

n	s_n , #1	s_n , #2
1	0.5000	1.0000
2	0.6250	0.5000
3	0.6667	0.8333
4	0.6823	0.5833

answer : #1 converges faster , why?



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<u>example</u> : evaluate $\int_0^{\frac{1}{2}} \frac{dx}{1+x^4}$

<u>method 1</u> : FTC

$$\int_0^{\frac{1}{2}} \frac{dx}{1+x^4} = \left[\frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \frac{\sqrt{2}}{4}(\tan^{-1}(\sqrt{2}x+1)+\tan^{-1}(\sqrt{2}x-1))\right]_0^{\frac{1}{2}}$$

<u>method 2</u> : Riemann sums

$$\int_0^{\frac{1}{2}} \frac{dx}{1+x^4} \approx \sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{2n}\right)^4}\cdot\frac{1}{2n}$$

$a = 0$, $b = \frac{1}{2}$, $\Delta x = \frac{b-a}{n} = \frac{1}{2n}$, $x_i = a + i\Delta x = \frac{i}{2n}$

<u>method 3</u> : series

$$\frac{1}{1+x^4} = \frac{1}{1-(-x^4)} = 1 - x^4 + x^8 - x^{12} + \cdots \text{ for } -1 < x < 1$$

recall : $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$ for $-1 < r < 1$

$$\int_0^{\frac{1}{2}} \frac{dx}{1+x^4} = \int_0^{\frac{1}{2}} \left(1 - x^4 + x^8 - x^{12} + \cdots\right) dx = \left(x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \cdots\right)\Big|_0^{\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{1}{5\cdot 2^5} + \frac{1}{9\cdot 2^9} - \frac{1}{13\cdot 2^{13}} + \cdots$$

$$\int_0^{\frac{1}{2}} \frac{dx}{1+x^4} \approx \frac{1}{2} - \frac{1}{5\cdot 2^5} = \frac{1}{2} - \frac{1}{160} = \frac{79}{160} = 0.49375$$

$$\left|\int_0^{\frac{1}{2}} \frac{dx}{1+x^4} - 0.49375\right| < \frac{1}{9\cdot 2^9} = \frac{1}{9\cdot 512} = \frac{1}{4608} < \frac{1}{4000} = 0.00025$$

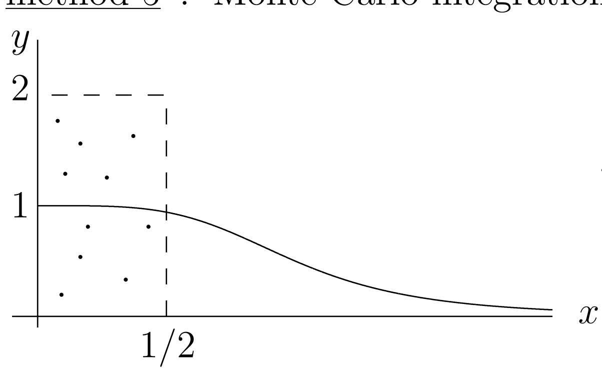
$$\Rightarrow 0.49350 < \int_0^{\frac{1}{2}} \frac{dx}{1+x^4} < 0.49400$$

<u>method 4</u> : Maple

> Int(1/(1+x^4),x=0..1/2); $\quad \int_0^{\frac{1}{2}} \left(\frac{1}{1+x^4}\right) dx$

> evalf(%); $\quad$ 0.4939580511 , which method does Maple use? ChatGPT?

<u>method 5</u> : Monte Carlo integration



$$\int_0^{\frac{1}{2}} \frac{dx}{1+x^4} = \frac{\text{area under graph}}{\text{area of box}}$$

$$\approx \frac{\text{\# points below graph}}{\text{total \# points in box}}$$

example

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ : \text{ converges for } -\infty < x < \infty$$

Can you identify $f(x)$?

$$f'(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow f'(x) = f(x)\,,\ f(0) = 1 \Rightarrow f(x) = e^x$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for } -\infty < x < \infty$$

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$$x = 1 \Rightarrow e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots = 2.7183$$

$$x = -1 \Rightarrow e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} - \cdots = 0.3679$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \text{ for } -\infty < x < \infty$$

$$\int e^{x^2} dx = x + \frac{x^3}{3} + \frac{x^5}{2 \cdot 5} + \frac{x^7}{3! \cdot 7} + \frac{x^9}{4! \cdot 9} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)} \text{ for } -\infty < x < \infty$$

3.7 Taylor series : $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, $c_n = ?$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

$$f(a) = c_0$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f'(a) = c_1$$

$$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \cdots$$

$$f''(a) = 2c_2 \ \Rightarrow \ c_2 = \frac{f''(a)}{2}$$

$$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + \cdots$$

$$f'''(a) = 2 \cdot 3c_3 \ \Rightarrow \ c_3 = \frac{f'''(a)}{3!}$$

$$\cdots$$

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdots n \cdot c_n \ \Rightarrow \ c_n = \frac{f^{(n)}(a)}{n!}$$

The Taylor series for $f(x)$ at $x = a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots.$$

<u>example 1</u> : $f(x) = \dfrac{1}{1-x}$, $a = 0$: find the Taylor series

$f(x) = (1-x)^{-1} \Rightarrow f(0) = 1$

$f'(x) = (1-x)^{-2} \Rightarrow f'(0) = 1$

$f''(x) = 2(1-x)^{-3} \Rightarrow f''(0) = 2$

$f'''(x) = 3 \cdot 2(1-x)^{-4} \Rightarrow f'''(0) = 3!$, ... , $f^{(n)}(0) = n!$

$$\text{TS} = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } -1 < x < 1$$

Hence the geometric series is a special case of a Taylor series.

<u>example 2</u> : $f(x) = \sin x$, $a = 0$: find the TS

$f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = 0$

$$\text{TS} = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

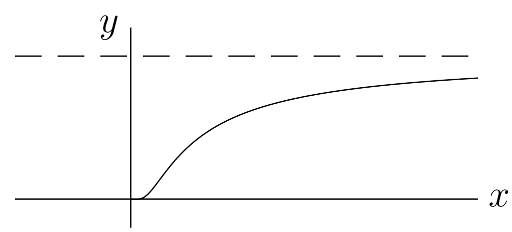
$$= \cancel{f(0)} + f'(0)x + \cancel{\frac{f''(0)}{2}}x^2 + \frac{f'''(0)}{3!}x^3 + \cancel{\frac{f^{(4)}(0)}{4!}}x^4 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \ : \ \text{converges for } -\infty < x < \infty$$

We still need to consider whether the TS converges to the given $f(x)$; sometimes it does, and sometimes it does not.

does : $\dfrac{1}{1-x}$, e^x , $\sin x$, $\cos x$, $J_0(x)$, ...

does not : $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$



$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{-1/h}}{h} = \frac{0}{0} = \lim_{t \to \infty} \frac{t}{e^t} = \frac{\infty}{\infty} = \lim_{t \to \infty} \frac{1}{e^t} = 0$$

In fact, $f^{(n)}(0) = 0$ for all n, hence the TS $= 0$, which converges for all x, but $f(x) \neq 0$ for $x > 0$.

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<u>recall</u>

hw7 : $f(x) = f(a) + \int_a^x f'(t)dt$

hw8 : $f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)dt$

hw9 : $f(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2}(x-a)^2 + \int_a^x \dfrac{(x-t)^2}{2} f'''(t)dt$

$\dots$

in general , $f(x) = T_n(x) + R_n(x)$

$T_n(x) = f(a) + f'(a)(x-a) + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$: nth partial sum of TS

$T_n(x)$: <u>Taylor polynomial</u> of degree n for $f(x)$ at $x = a$

$T_n(a) = f(a)\,,\ T_n'(a) = f'(a)\,,\ \dots\,,\ T_n^{(n)}(a) = f^{(n)}(a)$, but $T_n^{(n+1)}(a) = 0$

We view $T_n(x)$ as an <u>approximation</u> to $f(x)$ near $x = a$.

$R_n(x) = \int_a^x \dfrac{(x-t)^n}{n!} f^{(n+1)}(t)dt$: <u>remainder</u> , <u>error</u>

<u>error bound for Taylor approximation</u>

$|f(x) - T_n(x)| = |R_n(x)| \le \dfrac{M_{n+1}}{(n+1)!}|x-a|^{n+1}$, where $M_{n+1} = \max|f^{(n+1)}(t)|$

In many cases (but not all), the error decreases as n increases or $x \to a$.

<u>example</u>

$$\sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \text{ for all } x$$

$$\cos x \;=\; 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \text{ for all } x$$

<u>proof</u>

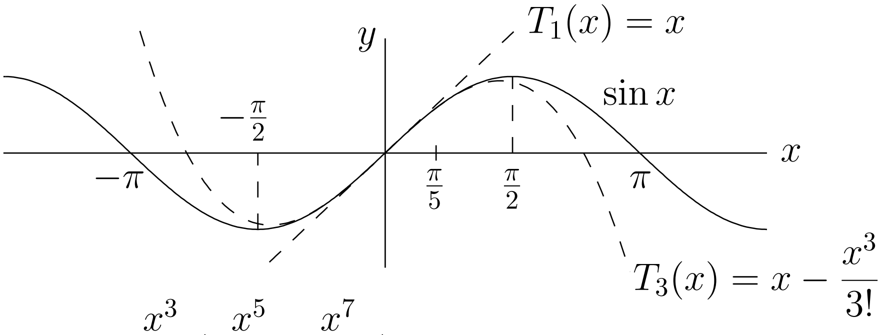
to show that $f(x) =$ TS, we must show that $f(x) = \lim\limits_{n\to\infty} T_n(x)$

or equivalently, that $|f(x) - T_n(x)| \to 0$ as $n \to \infty$ for all x

$$|f(x) - T_n(x)| = |R_n(x)| \le \frac{M_{n+1}}{(n+1)!}|x-a|^{n+1} = \frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty$$

$M_{n+1} = \max|f^{(n+1)}(t)| = 1$ <u>ok</u>

<u>example</u> : Taylor approximation for $\sin\frac{\pi}{5} = 0.58778525$



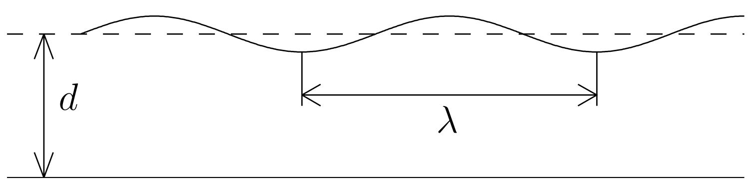
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\sin\frac{\pi}{5} \approx \frac{\pi}{5} = 0.6283 \Rightarrow |\sin\frac{\pi}{5} - \frac{\pi}{5}| < \frac{1}{3!}\left(\frac{\pi}{5}\right)^3 = 0.04134$, recall : $|s - s_n| < a_{n+1}$

$\sin\frac{\pi}{5} \approx \frac{\pi}{5} - \frac{1}{3!}\left(\frac{\pi}{5}\right)^3 = 0.5870 \Rightarrow \left|\sin\frac{\pi}{5} - \left[\frac{\pi}{5} - \frac{1}{3!}\left(\frac{\pi}{5}\right)^3\right]\right| < \frac{1}{5!}\left(\frac{\pi}{5}\right)^5 = 0.0008161$ 43 Tues 11/27

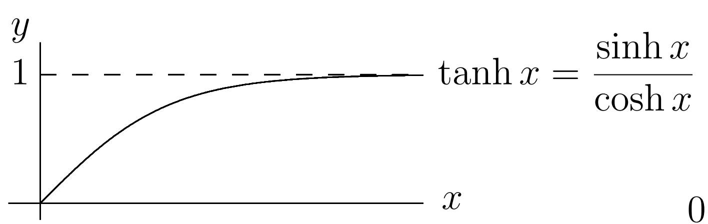
note : to compute $\sin\frac{4\pi}{5}$, consider Taylor approximation at $a = \pi$

<u>example</u> : water waves



d : depth
λ : wavelength
v : wave speed
g : acceleration due to gravity

given : $v^2 = \frac{g\lambda}{2\pi}\tanh\frac{2\pi d}{\lambda}$, two important cases : $\begin{cases} d/\lambda \to 0 & \text{: shallow water} \\ d/\lambda \to \infty & \text{: deep water} \end{cases}$



$\tanh x \sim \begin{cases} x \ \text{ for } \ x \to 0 \\ 1 \ \text{ for } \ x \to \infty \end{cases}$ proof : $\tanh x = \overset{0}{\cancel{\tanh(0)}} + \overset{1}{\cancel{\tanh'(0)}}x + \cdots$
proof : hw6

<u>shallow water</u> : $d/\lambda \to 0 \Rightarrow v^2 \sim \frac{g\cancel{\lambda}}{\cancel{2\pi}} \cdot \frac{\cancel{2\pi} d}{\cancel{\lambda}} = gd \Rightarrow v \sim (gd)^{1/2}$

$\Rightarrow$ In shallow water, the wave speed is independent of the wavelength.

<u>deep water</u> : $d/\lambda \to \infty \Rightarrow v^2 \sim \frac{g\lambda}{2\pi} \cdot 1 \Rightarrow v \sim \left(\frac{g\lambda}{2\pi}\right)^{1/2}$

$\Rightarrow$ In deep water, long waves travel faster than short waves.

l'Hôpital's rule

If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{f(a)}{g(a)} = \dfrac{0}{0}$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)}$.

proof : $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{\cancel{f(a)} + f'(a)\cancel{(x-a)} + \frac{1}{2}f''(a)(x-a)^{\cancel{2}} + \cdots}{\cancel{g(a)} + g'(a)\cancel{(x-a)} + \frac{1}{2}g''(a)(x-a)^{\cancel{2}} + \cdots} = \dfrac{f'(a)}{g'(a)}$ ok

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3.8 binomial series

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$f(x) = (1+x)^k \ \ , \ \ a = 0$

$$\text{TS} = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$f(x) = (1+x)^k \ \Rightarrow \ f(0) = 1$

$f'(x) = k(1+x)^{k-1} \ \Rightarrow \ f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2} \ \Rightarrow \ f''(0) = k(k-1)$

$f'''(x) = k(k-1)(k-2)(1+x)^{k-3} \ \Rightarrow \ f'''(0) = k(k-1)(k-2)$

$\ldots$

$$f^{(n)}(0) = \begin{cases} k(k-1)(k-2)\cdots(k-n+1) & \text{if } n \ge 1 \\ 1 & \text{if } n = 0 \end{cases}$$

theorem

$(1+x)^k = 1 + kx + \dfrac{k(k-1)}{2}x^2 + \dfrac{k(k-1)(k-2)}{3!}x^3 + \cdots$ for $-1 < x < 1$

For $x = \pm 1$, the series may converge or diverge depending on the value of k.

proof : ratio test , error bound ...

$$\begin{aligned}(1+x)^{-1} &= 1 + (-1)x + \frac{(-1)(-2)}{2}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots \ : \ \text{geometric series} \ , \ r = -x\end{aligned}$$

$$\begin{aligned}(1+x)^{-1/2} &= 1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3 + \cdots \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots\end{aligned}$$

$(1-x^2)^{-1/2} = (1+(-x^2))^{-1/2}$, replace $x \to -x^2$ in previous formula

$$\begin{aligned}&= 1 - \frac{1}{2}(-x^2) + \frac{3}{8}(-x^2)^2 - \frac{5}{16}(-x^2)^3 + \cdots \\ &= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots\end{aligned}$$

<u>example</u>: Einstein's special theory of relativity (1905)

m_0 : mass of an object at rest

v : velocity of object

m : mass of object when it's moving at velocity v

c : speed of light $= 3 \cdot 10^8$ m/s

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \Rightarrow \begin{cases} \text{if } v \to 0 \text{ , then } m \to m_0 \\ \text{if } v \to c \text{ , then } m \to \infty \end{cases}$$

kinetic energy = (total energy at velocity v) – (total energy at rest)

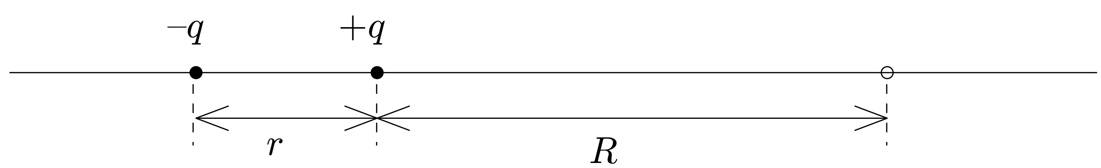
$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1 - v^2/c^2}} - m_0c^2 = m_0c^2\left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right)$$

assume $v << c$, set $x = \dfrac{v}{c} \Rightarrow \left(1 - \dfrac{v^2}{c^2}\right)^{-1/2} = \left(1 - x^2\right)^{-1/2} = 1 + \frac{1}{2}x^2 + \cdots$

$$K = m_0c^2\left(\cancel{1} + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \cdots - \cancel{1}\right) \approx m_0c^2 \cdot \frac{1}{2}\frac{v^2}{c^2} = \frac{1}{2}m_0v^2$$

$\Rightarrow$ Special relativity reduces to Newtonian mechanics for $v << c$.

<u>example</u>: Two ions with electric charge $+q$ and $-q$ form an <u>electric dipole</u>.



The electric field at the observation point is $E = \dfrac{+q}{4\pi\epsilon_0 R^2} + \dfrac{-q}{4\pi\epsilon_0 (R+r)^2}$.

This expression can be simplified for $R >> r$.

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$$E = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R^2} - \frac{1}{(R+r)^2}\right) = \frac{q}{4\pi\epsilon_0 R^2}\left(1 - \frac{1}{(1 + r/R)^2}\right)$$

$$= \frac{q}{4\pi\epsilon_0 R^2}\left(1 - \left(1 + \frac{r}{R}\right)^{-2}\right) \text{ , set } k = -2 \text{ , } x = \frac{r}{R} \Rightarrow x << 1$$

$$(1+x)^{-2} = 1 + (-2)x + \tfrac{(-2)(-3)}{2}x^2 + \cdots = 1 - 2x + 3x^2 - \cdots$$

$$E = \frac{q}{4\pi\epsilon_0 R^2}\left(\cancel{1} - \left(\cancel{1} - 2\frac{r}{R} + 3\frac{r^2}{R^2} - \cdots\right)\right) = \frac{q}{4\pi\epsilon_0 R^2}\left(2\frac{r}{R} - 3\frac{r^2}{R^2} + \cdots\right)$$

$\Rightarrow E \approx \dfrac{qr}{2\pi\epsilon_0 R^3}$: <u>far-field approximation</u> for the electric field of a dipole

qr = charge $\times$ distance = <u>electric dipole moment</u>

<u>recall</u>

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$
$$= 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

<u>special case</u> : From now on let k be a positive integer (i.e. $k = 1,\ 2,\ 3,\ \ldots$). Then $(1+x)^k$ is a polynomial of degree k and the Taylor series terminates at index $n = k$ because $f^{(n)}(0) = 0$ for $n \geq k+1$.

$$k = 1 \ : \ (1+x)^1 = 1 + 1x + \frac{1\cdot 0}{2}x^2 + \cdots = 1 + x$$

$$k = 2 \ : \ (1+x)^2 = 1 + 2x + \frac{2\cdot 1}{2}x^2 + \frac{2\cdot 1\cdot 0}{3!}x^3 + \cdots = 1 + 2x + x^2$$

<u>new notation</u>

$$\frac{k(k-1)\cdots(k-n+1)}{n!} \cdot \frac{(k-n)(k-n-1)\cdots 3\cdot 2\cdot 1}{(k-n)(k-n-1)\cdots 3\cdot 2\cdot 1}$$
$$= \frac{k!}{n!(k-n)!} = \binom{k}{n} \ : \ \underline{\text{binomial coefficient}}\ ,\ \underline{k \text{ choose } n}\ \text{(more later)}$$

note : $\displaystyle\binom{k}{0} = \frac{k!}{0!k!} = 1$

$$\Rightarrow \ (1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n = \sum_{n=0}^{k} \frac{k!}{n!\,(k-n)!}\, x^n \ \text{ for } \ k = 1, 2, 3, \ldots$$

$$k = 4 \ : \ (1+x)^4 = \sum_{n=0}^{4} \binom{4}{n} x^n = \binom{4}{0} + \binom{4}{1}x + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{4}x^4$$

$$\binom{4}{0} = \frac{4!}{0!\,4!} = 1 \quad , \quad \binom{4}{1} = \frac{4!}{1!\,3!} = 4 \quad , \quad \binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

$$\binom{4}{3} = \frac{4!}{3!\,1!} = 4 \quad , \quad \binom{4}{4} = \frac{4!}{4!\,0!} = 1 \ \Rightarrow (1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

<u>alternative</u>

$$(1+x)^4 = (1+x)(1+x)(1+x)(1+x)$$
$$= 1 + x(1+1+1+1) + x^2(1+1+1+1+1+1) + x^3(1+1+1+1) + x^4$$
$$= 1 + 4x + 6x^2 + 4x^3 + x^4$$

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<u>combinatorial interpretation</u> of binomial coefficients

recall : $k = 1, 2, 3, \ldots$

$(1+x)^k = (1+x)(1+x)\cdots(1+x)$: k factors

$$= \binom{k}{0} + \binom{k}{1}x + \binom{k}{2}x^2 + \cdots + \binom{k}{n}x^n + \cdots + \binom{k}{k}x^k = \sum_{n=0}^{k}\binom{k}{n}x^k$$

$\displaystyle\binom{k}{n} = \frac{k!}{n!(k-n)!} = \frac{f^{(n)}(0)}{n!}$, where $f(x) = (1+x)^k$

$=$ coefficient of x^n

$=$ number of ways of choosing n ~~factors~~ objects from a set of k ~~factors~~ objects, disregarding the order in which they are chosen

$=$ <u>k choose n</u>

<u>example</u> : $k = 4$, $n = 2 \Rightarrow 4$ objects : $\{A, B, C, D\}$, $\displaystyle\binom{4}{2} = \frac{4!}{2!\,2!} = 6$

$\Rightarrow$ there are 6 ways of choosing 2 objects from a set of 4 objects, dtoiwtac

$\{AB, AC, AD, BC, BD, CD\}$, note : BA is the same as AB

<u>theorem</u> : $\displaystyle(a+b)^k = \sum_{n=0}^{k}\binom{k}{n}a^{k-n}b^n$: <u>binomial formula</u>

<u>proof</u>

$$(a+b)^k = \left(a\left(1+\frac{b}{a}\right)\right)^k = a^k\left(1+\frac{b}{a}\right)^k = a^k\sum_{n=0}^{k}\binom{k}{n}\left(\frac{b}{a}\right)^n = \sum_{n=0}^{k}\binom{k}{n}a^{k-n}b^n \quad \underline{\text{ok}}$$

<u>example</u> : $\displaystyle(a+b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$

$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

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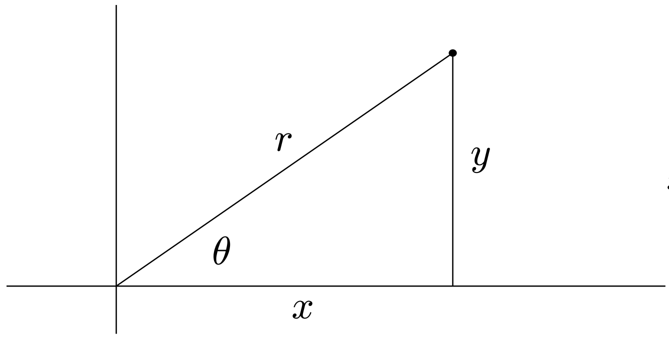
4 miscellaneous

4.1 polar coordinates

(x,y) : Cartesian coordinates , (r,θ) : polar coordinates



$$r=\sqrt{x^2+y^2}\ ,\ \theta=\tan^{-1}\frac{y}{x}$$
$$x=r\cos\theta\ ,\ y=r\sin\theta$$

In 3D the most important coordinate systems are Cartesian (x,y,z), cylindrical (r,θ,z), and spherical (ρ,ϕ,θ). Math 215, 285

4.2 complex numbers

$ax^2+bx+c=0 \Rightarrow x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$: quadratic formula

example : $x^2-2x+2=0$

$x^2-2x+2=x^2-2x+1+1=(x-1)^2+1>0 \Rightarrow$ no real solutions

$$x=\frac{2\pm\sqrt{4-4\cdot 2}}{2}=1\pm\frac{\sqrt{-4}}{2}=1\pm\frac{\sqrt{-1}\sqrt{4}}{2}=1\pm i\ ,\ i=\sqrt{-1}$$

proof : complete the square

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$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)=a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right)$$

$$=a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right)=0\ ,\ \text{assume } a\neq 0$$

$$\Rightarrow \left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}\Rightarrow x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}\qquad \underline{\text{ok}}$$

A complex number has the form $z=x+iy$, where x,y are real and $i=\sqrt{-1}$.

x : real part , y : imaginary part , $\overline{z}=x-iy$: complex conjugate

arithmetic : $z_1+z_2\ ,\ z_1-z_2\ ,\ z_1z_2\ ,\ \dfrac{z_1}{z_2}$

examples : $(1+i)+(1-i)=2$

$$(1+i)-(1-i)=2i$$

$$(1+i)(1-i)=1-i+i-i^2=2$$

$$\frac{1+i}{1-i}=\frac{1+i}{1-i}\cdot\frac{1+i}{1+i}=\frac{1+2i+i^2}{2}=i$$

<u>Euler's formula</u> : $e^{i\theta} = \cos\theta + i\sin\theta$

<u>proof</u> : $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \cdots$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = \cos\theta + i\sin\theta \qquad \underline{\text{ok}}$$

<u>examples</u>

$e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$	$e^{\pi i/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$
$e^{\pi i} = \cos\pi + i\sin\pi = -1$	$e^{\pi i/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$
$e^{\pi i/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$	$e^{\pi i/6} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$

A complex number $z = x + iy$ defines a point in the xy-plane using (x, y) as Cartesian coordinates, and it can also be written using polar coordinates, $z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

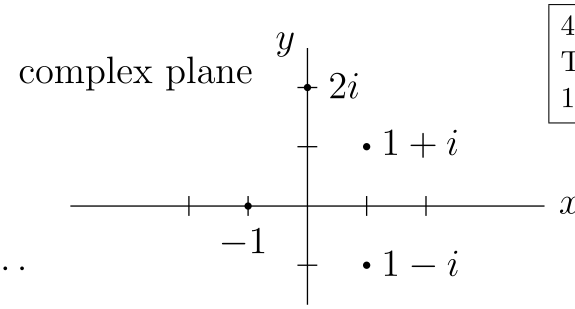
<u>examples</u> 48 Tues 12/3

$z = 2i = 2e^{\pi i/2}$

$z = 1 + i = \sqrt{2}\, e^{\pi i/4}$

$z = 1 - i = \sqrt{2}\, e^{-\pi i/4} = \sqrt{2}\, e^{7\pi i/4} = \ldots$

$z = -1 = e^{\pi i} \Rightarrow \log(-1) = \pi i\,,\ -\pi i\,,\ 3\pi i\,,\ \ldots$



$\uparrow$

<u>complex logarithm</u> : $\log z = \log re^{i\theta} = \log r + \log e^{i\theta} = \ln r + i\theta$

<u>problem</u> : Find all solutions of $z^3 = 1$.

This is a cubic polynomial equation, so there are 3 solutions.

$$z^3 - 1 = (z-1)(z^2+z+1) = 0 \Rightarrow \begin{cases} z - 1 = 0 \Rightarrow z = 1 \\ z^2 + z + 1 = 0 \Rightarrow z = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \end{cases}$$

alternative

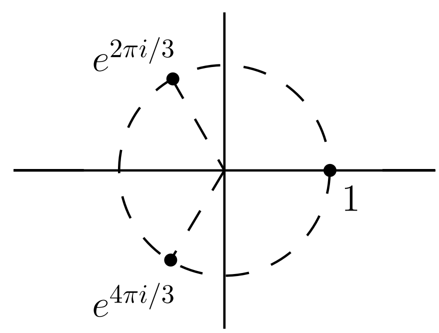
$$z = re^{i\theta} \Rightarrow z^3 = r^3 e^{3i\theta} = 1 \Rightarrow \begin{cases} r^3 = 1 \Rightarrow r = 1 \\ 3\theta = 0, 2\pi, 4\pi, 6\pi, \ldots \end{cases}$$

$\theta_1 = 0 \Rightarrow z_1 = e^0 = 1$

$\theta_2 = \frac{2\pi}{3} \Rightarrow z_2 = e^{2\pi i/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$\theta_3 = \frac{4\pi}{3} \Rightarrow z_3 = e^{4\pi i/3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\theta_4 = 2\pi \Rightarrow z_4 = e^{2\pi i} = 1 = z_1\,,\ \ldots$



<u>double-angle formulas</u> : $\cos 2x = \cos^2 x - \sin^2 x$, $\sin 2x = 2\sin x \cos x$

<u>proof</u>

$(e^{ix})^2 = e^{2ix} = \cos 2x + i\sin 2x$

$(e^{ix})^2 = (\cos x + i\sin x)^2 = \cos^2 x - \sin^2 x + 2i\sin x\cos x$ <u>ok</u>

<u>recall</u> : $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$, $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

<u>proof</u> : use $\cos^2 x + \sin^2 x = 1$

$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$

$\Rightarrow 2\cos^2 x = 1 + \cos 2x , \ \ldots$ <u>ok</u>

$\cos 2x = \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$

$\Rightarrow 2\sin^2 x = 1 - \cos 2x , \ \ldots$ <u>ok</u>

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proof of $\cos^2 x + \sin^2 x = 1$ using complex numbers 49 Wed 12/4

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i\sin x$

$e^{ix} \cdot e^{-ix} = (\cos x + i\sin x)(\cos x - i\sin x)$

$= \cos^2 x + \cancel{\cos x \cdot -i\sin x} + \cancel{i\sin x \cdot \cos x} - i^2\sin^2 x = \cos^2 x + \sin^2 x$

$e^{ix} \cdot e^{-ix} = e^{ix - ix} = e^0 = 1$ <u>ok</u>

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proof of $\cos^2 x + \sin^2 x = 1$ using power series

$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$

$\cos^2 x = (1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots)(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots)$

$= 1 + x^2\left(-\frac{1}{2} - \frac{1}{2}\right) + x^4(\frac{1}{4!} + \frac{1}{4} + \frac{1}{4!}) + x^6(-\frac{1}{6!} - \frac{1}{2\cdot 4!} - \frac{1}{2\cdot 4!} - \frac{1}{6!}) + \cdots$

$= 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \cdots$

$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$

$\sin^2 x = (x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots)(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots)$

$= x^2 + x^4(-\frac{1}{3!} - \frac{1}{3!}) + x^6(\frac{1}{5!} + \frac{1}{3!\cdot 3!} + \frac{1}{5!}) + \cdots$

$= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \cdots$

$\cos^2 x + \sin^2 x = 1$ <u>ok</u>

$\cos^2 x - \sin^2 x = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots$

$= 1 - \frac{1}{2}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6 + \cdots = \cos 2x$ ok

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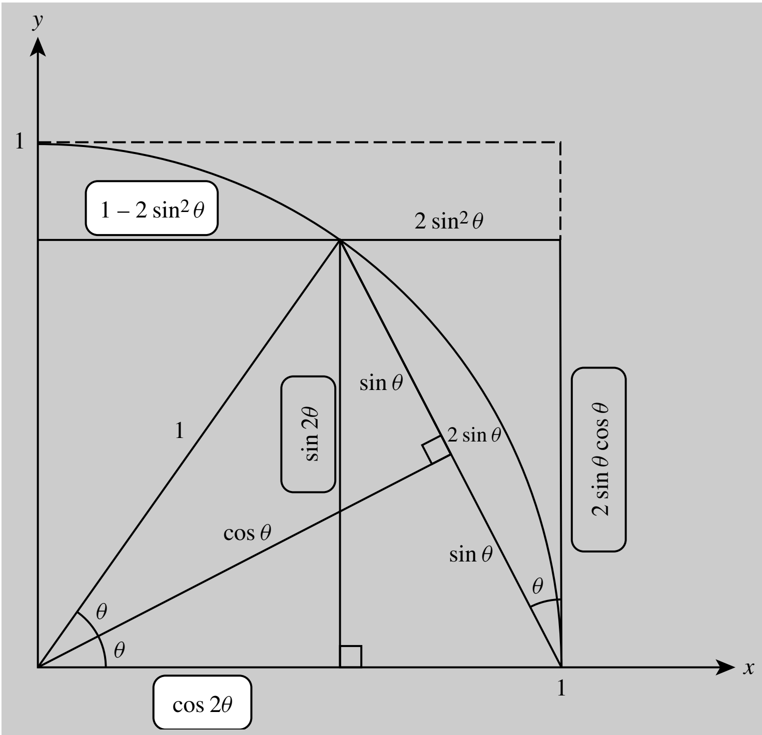
"Proof Without Words" by Hasan Unal (Yildiz Technical University, Turkey)
College Mathematics Journal, November 2010, page 392

$\cos 2\theta = 1 - 2\sin^2\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$



<u>addition formulas</u>

$\sin(a + b) = \sin a\ \cos b + \cos a\ \sin b$

$\cos(a + b) = \cos a\ \cos b - \sin a\ \sin b$

<u>proof</u>

$e^{i(a+b)} = \cos(a + b) + i\sin(a + b)$

$e^{i(a+b)} = e^{ia} \cdot e^{ib} = (\cos a + i\sin a)(\cos b + i\sin b)$

$\qquad = (\cos a\cos b - \sin a\sin b) + i(\cos a\sin b + \sin a\cos b)$ <u>ok</u>
