Thanksgiving Class

example 1

recall : $y(t)\,,\ \dfrac{dy}{dt} = f(y)$: ODE

now consider : $y(x,t)$, define $\dfrac{\partial y}{\partial t}\,,\ \dfrac{\partial y}{\partial x}$: partial derivatives

$\dfrac{\partial y}{\partial t} = \dfrac{\partial^2 y}{\partial x^2}$: heat equation : PDE

$y(x,t) = e^{-t}\sin x$, check ...

$y(x,t) = e^{-n^2 t}\sin nx$, n : wavenumber

$y(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t}\sin nx$: Fourier series , Math 354, 454

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example 2

theorem : $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$

proof (Euler's solution of the Basel problem)

$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$: Taylor series

$$\Rightarrow \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

note : $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1\,,\ \frac{\sin x}{x} = 0$ for $x = \pm\pi\,,\ \pm 2\pi\,,\ \pm 3\pi\,,\ \ldots$

$$\Rightarrow \frac{\sin x}{x} = \Big(1 - \frac{x^2}{\pi^2}\Big)\Big(1 - \frac{x^2}{4\pi^2}\Big)\Big(1 - \frac{x^2}{9\pi^2}\Big)\cdots$$

$$= 1 - x^2\Big(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\Big) + \cdots$$

$$\Rightarrow \frac{1}{3!} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots = \frac{1}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots\right)$$ ok

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<u>example 3</u> , recall : alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2 \text{ (page 64)}$$

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \cdots = \frac{1}{2}\ln 2$$

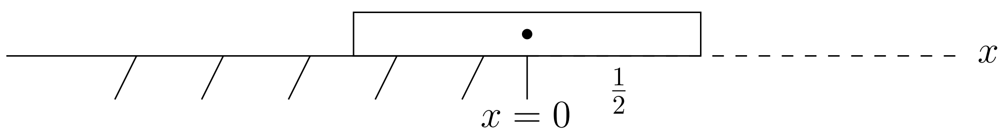
$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots = \frac{1}{2}\ln 2$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots\cdots\cdots\cdots\cdots = \frac{3}{2}\ln 2$$

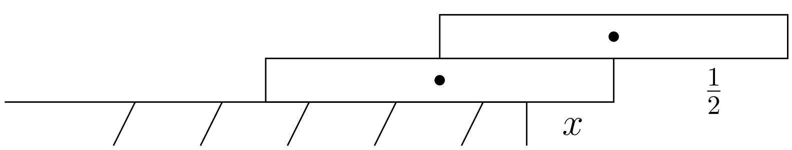
Hence the sum depends on the order in which the terms are added.

<u>example 4</u> : Consider an infinite supply of identical books of length 1.

How far can we extend 1 book past the edge of a table?



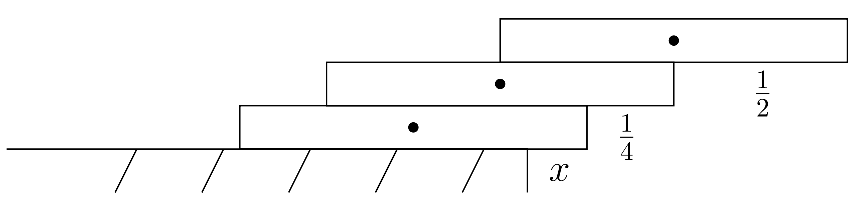
$\cdots\cdots$ 2 books $\cdots\cdots$?



Let x_i be the coordinate of the CM of the ith book.

$$\bar{x} = \frac{M}{m} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{x_1 + x_2}{2} = 0 \ , \ \left.\begin{array}{l} x_1 = x \\ x_2 = x - \frac{1}{2} \end{array}\right\} \Rightarrow 2x - \frac{1}{2} = 0 \Rightarrow x = \frac{1}{4}$$

$\cdots\cdots$ 3 books $\cdots\cdots$?



$$x_1 + x_2 + x_3 = 0 \ , \ \left.\begin{array}{l} x_1 = x + \frac{1}{4} \\ x_2 = x + \frac{1}{4} - \frac{1}{2} = x - \frac{1}{4} \\ x_3 = x - \frac{1}{2} \end{array}\right\} \Rightarrow 3x - \frac{1}{2} = 0 \Rightarrow x = \frac{1}{6}$$

infinitely many books $\Rightarrow$ distance $= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots = \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3} + \cdots) = \infty$