chapter 7 : time-dependent differential equations

ordinary differential equations

Let $x(t)$ be the position of a particle moving on the x-axis at time t.

1st order ODE

$\frac{dx}{dt} = f(x)$: velocity is a function of position

$x(0)$: initial position

The problem is to find the position $x(t)$ for $t > 0$.

ex

1. $\frac{dx}{dt} = x \; , \; x(0) = 1 \;\Rightarrow\; x(t) = e^t$

2. $\frac{dx}{dt} = x^2 \; , \; x(0) = 1 \;\Rightarrow\; x(t) = \frac{1}{1-t}$

3. $\frac{dx}{dt} = \sin x \; , \; x(0) = 1 \;\Rightarrow\; x(t) = \,?$

26 Thurs 4/18

The simplest numerical method is Euler's method.

choose Δt : time step

define w_n : numerical solution at time $t_n = n\Delta t$

$$\frac{w_{n+1} - w_n}{\Delta t} = f(w_n) \;\Rightarrow\; w_{n+1} = w_n + \Delta t f(w_n)$$

given w_0, compute $w_1, w_2, \ldots$

questions : accuracy , stability , efficiency

2nd order ODE

$\frac{d^2x}{dt^2} = f(x)$: acceleration is a function of position (Newton's equation)

$x(0)$, $x'(0)$: initial position , velocity

$$\frac{w_{n+1} - 2w_n + w_{n-1}}{(\Delta t)^2} = f(w_n) \;\Rightarrow\; w_{n+1} = 2w_n - w_{n-1} + (\Delta t)^2 f(w_n)$$

given w_0 and w_1, compute $w_2, w_3, \ldots$

partial differential equations

ex : heat equation

$u(x,t)$: temperature of a metal rod at position x and time t

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \quad , \quad \kappa \ : \text{ coefficient of thermal expansion}$$

initial condition : $u(x,0) = f(x)$, boundary conditions : $u(0,t) = u(1,t) = 0$

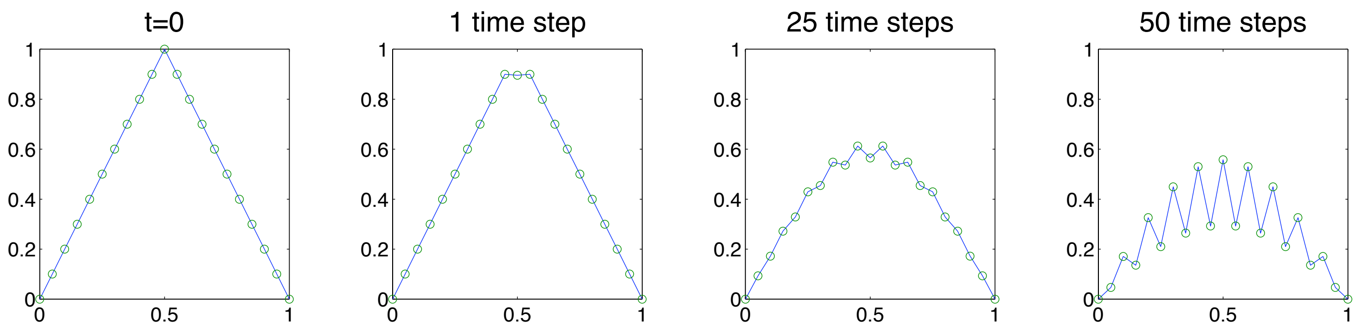
The simplest numerical method is a finite-difference scheme.

choose Δx : space step , Δt : time step

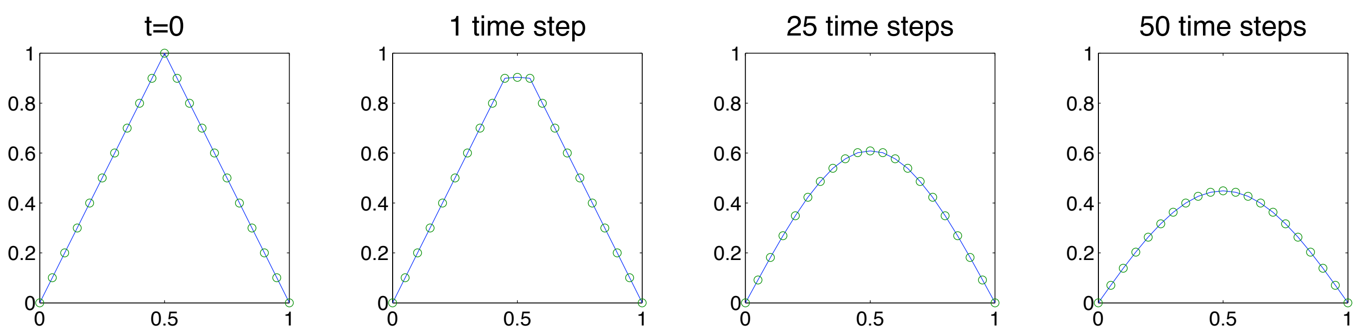
define w_j^n : numerical solution at position $x_j = j\Delta x$ and time $t_n = n\Delta t$

$$\frac{w_j^{n+1} - w_j^n}{\Delta t} = \kappa \frac{w_{j+1}^n - 2w_j^n + w_{j-1}^n}{(\Delta x)^2} \Rightarrow w_j^{n+1} = w_j^n + \frac{\kappa \Delta t}{(\Delta x)^2}(w_{j+1}^n - 2w_j^n + w_{j-1}^n)$$

case 1 : $\kappa = 1$, $\Delta x = 0.05$, $\Delta t = 0.0013$



case 2 : $\kappa = 1$, $\Delta x = 0.05$, $\Delta t = 0.0012$



explanation : the method is stable $\Leftrightarrow \dfrac{\kappa \Delta t}{(\Delta x)^2} \leq \dfrac{1}{2}$ 27 Tues 4/23

Lax equivalence theorem (Peter Lax)

Given a well-posed initial/boundary value problem and a consistent finite-difference scheme, stability is necessary and sufficient for convergence.

Fourier analysis

differential equation

$u(x,t) = e^{\omega t + ikx}$: Fourier mode , ω : growth rate , k : wavenumber

$\Rightarrow \omega = -k^2 \ \Rightarrow \ u(x,t) = e^{-k^2 t + ikx}$

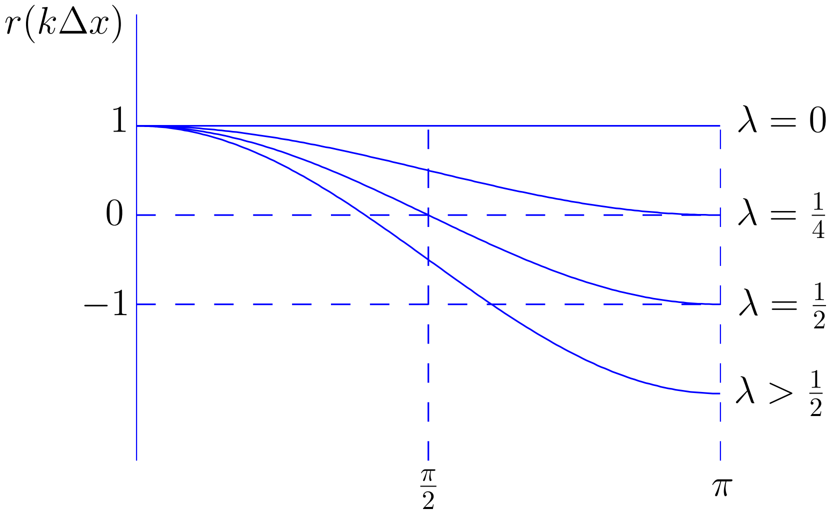
$\Rightarrow$ all modes with $k \neq 0$ decay monotonically in time

finite-difference scheme

$w_j^n = r^n e^{ikj\Delta x}$: discrete Fourier mode , r : amplification factor

$$r^{n+1} e^{ikj\Delta x} = r^n e^{ikj\Delta x} + \lambda \left(r^n e^{ik(j+1)\Delta x} - 2r^n e^{ikj\Delta x} + r^n e^{ik(j-1)\Delta x} \right) \ , \ \lambda = \frac{\kappa \Delta t}{(\Delta x)^2}$$

$$\Rightarrow r = 1 + \lambda \left(e^{ik\Delta x} - 2 + e^{-ik\Delta x} \right) = 1 + 2\lambda(\cos k\Delta x - 1) = 1 - 4\lambda \sin^2 \tfrac{1}{2} k\Delta x$$



note

1. $|r(k\Delta x))| \leq 1$ for all $k\Delta x \Leftrightarrow \lambda \leq \frac{1}{2}$

2. $|w_j^n| = |r|^n$

$0 \leq \lambda \leq \frac{1}{4}$: all modes decay monotonically in time

$\frac{1}{4} \leq \lambda \leq \frac{1}{2}$: $\begin{cases} \text{long waves decay monotonically in time} \\ \text{short waves oscillate in sign, amplitude decays} \end{cases}$

$\lambda > \frac{1}{2}$: $\begin{cases} \text{long waves decay monotonically in time} \\ \text{intermediate waves oscillate in sign, amplitude decays} \\ \text{short waves oscillate in sign, amplitude grows} \end{cases}$

This is in sharp contrast with the PDE.

<u>ex</u> : wave equation

$u(x,t)$: displacement of a string at position x and time t

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

c : wave speed

$$\frac{w_i^{n+1} - 2w_i^n + w_i^n}{(\Delta t)^2} = c^2 \frac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{(\Delta x)^2}$$

$$\Rightarrow \; w_i^{n+1} = 2w_i^n - w_i^{n-1} + \left(\frac{c\Delta t}{\Delta x}\right)^2 (w_{i+1}^n - 2w_i^n + w_{i-1}^n) \quad , \; \text{stable} \Leftrightarrow \left|\frac{c\Delta t}{\Delta x}\right| \leq 1$$