Math 371 Review Sheet for Final Exam Winter 2013

The final exam is on Wednesday, May 1, 1:30-3:30pm. Section 1 (Krasny) will meet in 1017 DOW and Section 2 (Wang) will meet in 1005 DOW. The exam will cover the entire course. You may use a calculator to do arithmetic. Please explain your steps to receive full credit. You may use two sheets of notes. Exam booklets will be provided. All vector and matrix norms are the infinity-norm.

1. True or False? Give a reason to justify your answer.

a) $D_+D_-y_i = D_-D_+y_i$

b) $D_0 f(x)$ is a 2nd order accurate approximation for the 2nd derivative $f''(x)$.

c) If $Ax = 0$, then $A = 0$ or $x = 0$.

d) If A is invertible, then $||A^{-1}|| = ||A||$.

e) $\rho(B) \le ||B||$ for any matrix B

f) The spectral radius of a matrix satisfies the properties required to be a matrix norm.

g) In solving an $n \times n$ system of linear equations by Gaussian elimination, if n increases by a factor of 3, then the operation count increases by a factor of approximately 9.

h) In computing the solution of a linear system $Ax = b$, if the residual norm $||r||$ is small, then the error norm $||e||$ is also small.

i) If $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, then Jacobi's method applied to solve $Ax = b$ converges.

j) In solving the linear system $Ax = b$, where $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, one step of Gauss-Seidel reduces the norm of the error as much as two steps of Jacobi.

k) In solving a linear system $Ax = b$ by an iterative method $x_{k+1} = Bx_k + c$, if $||B|| < 1$, then $\lim_{k\to\infty} x_k = x$ for any initial guess x_0.

l) In solving a linear system $Ax = b$ by an iterative method such as Jacobi or Gauss-Seidel, if the matrix A has dimension $n \times n$, then the exact solution is obtained after n iterations.

m) Consider two iterative methods for solving $Ax = b$. If the two iteration matrices B_1, B_2 satisfy $||B_1|| = ||B_2||$, then the two iterative methods converge at the same rate.

n) Suppose a two-dimensional boundary value problem is solved using a finite-difference scheme and the resulting linear system is solved by Jacobi's method with stopping criterion $||r_k|| \le 10^{-2}$. If the mesh size h is decreased, then the number of iterations needed to satisfy the stopping criterion is also decreased.

o) Jacobi and Gauss-Seidel converge linearly, but optimal SOR converges quadratically.

p) A matrix A is positive definite if there exists at least one vector $x \ne 0$ such that $x^TAx > 0$.

q) If $\lambda = 0$ is an eigenvalue of A, then A is not invertible.

r) If A is symmetric and positive definite, then A is invertible.

s) The inverse power method is used to compute the inverse of a matrix.

t) Wilkinson's example shows that the coefficients of a polynomial can depend sensitively on the roots.

u) When the power method is applied to find the largest eigenvalue and corresponding eigenvector of a matrix, the vectors $v^{(k)}$ are normalized at each step in order to accelerate convergence of the method.

v) If $p_n(x)$ is the interpolating polynomial of degree n for a given function $f(x)$ at points $x_i = a + ih$, where $h = \frac{b-a}{n}$ and $i = 0 : n$, then $\lim_{n\to\infty} p_n(x) = f(x)$ for all x in the interval $[a, b]$.

w) Polynomial interpolation at the Chebyshev points on the interval $-1 \le x \le 1$ gives a good approximation near the center of the interval and a bad approximation near the endpoints.

x) Suppose $f(x)$ is approximated by a cubic spline interpolant $s(x)$ on the interval $a \le x \le b$ with interpolation points $x_i = a + ih$, where $h = \frac{b-a}{n}$ and $i = 0 : n$. Then if n is doubled, the error defined by $\max_{a\le x\le b} |f(x) - s(x)|$ is reduced by a factor of approximately $1/16$.

2. State one advantage of ...
a) ... Newton's method over the bisection method.
b) ... Gaussian elimination with pivoting over Gaussian elimination without pivoting.
c) ... optimal SOR over Gauss-Seidel.
d) ... Chebyshev points over uniform points.
e) ... cubic spline interpolation over Taylor approximation.

3. Consider the following approximation for the first derivative, $f'(x) \approx \frac{-3f(x)+4f(x+h)-f(x+2h)}{2h}$.

a) Apply the method to compute $f'(1)$, for $f(x) = e^x$, with step size $h = 1, \frac{1}{2}, \frac{1}{4}$. Present the results in a table with the following format. column 1: h (step size), column 2: approximation, column 3: error, column 4: error/h, column 5: error/h^2.

b) The error has the form: error $= cf^{(m)}(x)h^n + \cdots$. Find the constants c, m, n by Taylor expansion. Are the results of parts (a) and (b) consistent? Explain.

4. Consider solving the equation $f(x) = x^2 - 5 = 0$ by the bisection method.
a) Explain why $0 \le x \le 4$ is a suitable starting interval.
b) Take 3 steps of the bisection method, i.e. compute x_0, x_1, x_2.
c) Approximately how many steps are needed to ensure that the error is less than 10^{-4}?

5. Suppose fixed-point iteration is applied to the function $g(x) = x^2 - \frac{1}{2}x + \frac{1}{2}$. Find the fixed points and in each case determine whether the iteration converges for starting values sufficiently close to the fixed point.

6. The component form of SOR for a 2×2 system is given below. Correct any errors.

$$\begin{aligned} a_{11}x_1^{(k+1)} &= a_{11}x_1^{(k)} - \omega(a_{11}x_1^{(k)} + a_{12}x_2^{(k)} - b_1) \\ a_{22}x_2^{(k+1)} &= a_{22}x_2^{(k)} - \omega(a_{21}x_1^{(k+1)} + a_{22}x_2^{(k+1)} - b_2) \end{aligned}$$

7. Consider the linear system $2x_1 - x_2 = 1, -x_1 + 2x_2 - x_3 = 0, -x_2 + 2x_3 = 1$, with solution $x_1 = x_2 = x_3 = 1$. a) Write out Jacobi's method in component form. Take one step starting from the zero vector. Compute the error norms $||e_0||, ||e_1||$. b) Repeat for Gauss-Seidel.

8. Let $A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

a) For which of these does Jacobi's method converge?
b) For which of these does Gauss-Seidel converge?

9. Which of the following matrices are positive definite? a) $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, b) $\begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix}$

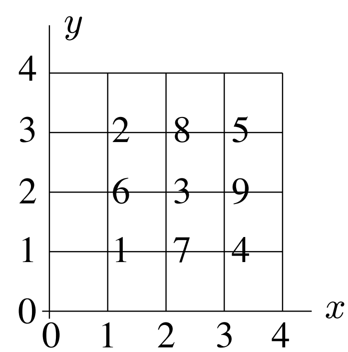
10. Let $A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}$.

a) Find a vector x such that $||Ax|| = ||A||$.
b) If $Ax = b$ is solved by Jacobi's method, does the iteration converge for any initial guess?
c) Answer the same question for Gauss-Seidel.
d) Find the optimal SOR parameter ω_*.
In parts (e) and (f) below, take $v^{(0)} = \frac{1}{\sqrt{3}}(1,1,1)^T$.
e) Find an approximate e-value of A by taking one step of the power method.
f) Find an approximate e-value of A by taking one step of the inverse power method.

11. Prove the following results or give a counterexample.
a) If A is positive definite, then A is invertible.
b) If A is positive definite, then the diagonal elements of A are positive.
c) If A is positive definite, then the eigenvalues of A are positive.
d) If A is invertible, then A^TA is symmetric and positive definite.

12. The 2-point boundary value problem $-y'' + y = x, y(0) = 1, y(1) = 0$ for $0 \le x \le 1$ is solved by the finite-difference scheme $-D_+D_-w_i + w_i = x_i$ for $i = 1 : n$, with step size $h = 1/(n+1)$, where $x_i = ih$ and $w_0 = 1, w_{n+1} = 0$. Using $n = 3$, write down the linear system $A_hw_h = f_h$.

13. Consider the Poisson equation $-\Delta\phi = f$ with boundary condition $\phi = g$, on the unit square $0 \le x, y \le 1$. Let the domain be discretized with mesh size $h = \frac{1}{4}$. Then there are nine unknown values in the interior of the domain, w_{ij}, for $i, j = 1, 2, 3$. Suppose the equation is discretized using the finite-difference scheme discussed in class, $-(D_+^xD_-^xw_{ij} + D_+^yD_-^yw_{ij}) = f_{ij}$, and the linear system is written as $A_hw_h = f_h$, where the elements of w_h have the ordering shown in the figure (this is called red-black ordering, like a checkerboard, and it is different than the ordering considered in class). Write down the matrix A_h in this case.



14. Apply the spectral method (see hw8) to solve $Ax = b$, where $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

15. Determine whether $\rho(A) = ||A||$. a) $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$ c) $\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$

16. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Find $\max_{x \ne 0} R_A(x)$ and $\min_{x \ne 0} R_A(x)$. Make sure to justify your answer.

17. Construct the divided difference table for the following data and find the Newton form of the interpolating polynomial $p_2(x)$. $(x_0, x_1, x_2) = (2, 4, 5), (y_0, y_1, y_2) = (-1, 4, 8)$. Use the polynomial to estimate the missing data for $x = 3$.

18. Let $f(x) = 1/x$ and take $(x_0, x_1, x_2, x_3) = (1, 2, 3, 4)$. Find the Newton form of the interpolating polynomial $p_3(x)$. Verify that $p_3(x)$ interpolates $f(x)$ at the given points.

19. The outdoor temperature $T(t)$ is recorded at two-hour intervals starting at 8am and ending at 4pm, but the 12pm measurement was accidentally omitted. The recorded temperatures are $T(\text{8am}) = 30°\,\text{F}$, $T(\text{10am}) = 40°\,\text{F}$, $T(\text{2pm}) = 50°\,\text{F}$, $T(\text{4pm}) = 60°\,\text{F}$. Use a cubic interpolating polynomial to estimate the missing temperature.

20. The thermal conductivity of air as a function of temperature is given in the table below. estimate the thermal conductivity of air when $T = 240\text{K}$ using the Newton form of the interpolating polynomial.

temperature (K)	100	200	300	400	500	600
thermal conductivity (mW/m · K)	9.4	18.4	26.2	33.3	39.7	45.7

21. Determine whether $s(x)$ is a natural cubic spline.

a) $s(x) = \begin{cases} 0 & , 0 \le x \le 1 \\ x^3 - 3x^2 + 3x - 1 & , 1 \le x \le 2 \end{cases}$ b) $s(x) = \begin{cases} -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & , -1 \le x \le 0 \\ \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & , \quad 0 \le x \le 1 \end{cases}$

22. Consider the natural cubic spline $s(x)$ satisfying $s(0) = 0, s(1) = 1, s(2) = 0$. a) Find the value of $s''(1)$. b) Sketch the graph of $s(x)$ on the interval $0 \le x \le 2$.

23. Compute the integral $\int_0^1 \sin \pi x dx$ using the trapezoid rule and present the results in a table with the following columns. column 1: h (take $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$), column 2: $T(h)$, column 3: $|I - T(h)|$, column 4: $|I - T(h)|/h$, column 5: $|I - T(h)|/h^2$, where I is the exact value and $T(h)$ is given by the trapezoid rule. What is the order of accuracy? Explain.

b) Repeat for $\int_0^1 \sqrt{x} dx$. What is the order of accuracy in this case? Explain.

24. Consider the formula $\int_0^{2h} f(x)dx \approx c_0 f(0) + c_1 f(h) + c_2 f(2h)$, where c_0, c_1, c_2 are constants.
a) Find values of c_0, c_1, c_2 which ensure that the formula is exact for $f(x) = 1, x, x^2$.
b) Is the formula exact for $f(x) = x^3, x^4, x^5$? (note: this formula is called Simpson's rule.)

25. Consider the integral $I = \int_0^1 xe^{-x^2} dx$.
a) Compute I using the trapezoid rule and Richardson extrapolation with $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.
b) Estimate how small h must be for the trapezoid rule to give the same error as in the last column of the extrapolation table.

26. Among the functions $\{1, x, x^2, \sin \pi x, \cos \pi x, \sin 2\pi x, \sin^2 \pi x\}$, find all pairs that are orthogonal with respect to the inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$.

27. The Legendre polynomials $P_0(x) = 1, P_1(x) = x, P_2(x) = x^2 - \frac{1}{3}, P_3(x) = x^3 - \frac{3}{5}x$ were derived in class. Now apply the Gram-Schmidt process to find $P_4(x)$. Verify that your expression for $P_4(x)$ is orthogonal to $P_i(x), i = 0 : 3$.

28. Evaluate the integral $\int_0^1 e^{-x} \sin \pi x \, dx$ by the following methods.
a) right-hand Riemann sum with $h = 1, \frac{1}{2}, \frac{1}{4}$
b) Richardson extrapolation applied to the results in (a)
c) trapezoid rule with $h = 1, \frac{1}{2}, \frac{1}{4}$
d) Richardson extrapolation applied to the results in (c)
e) three-point Gaussian quadrature
f) Which method is the most accurate? (hint: find the exact value using the methods of calculus, e.g. integration by parts, FTC, etc.)