Math 371 Winter 2013 Homework 4 due: Tuesday February 19

Solve the problems by hand, but you may use Matlab or a calculator to do arithmetic or check your answers. All vector norms and matrix norms are the ∞-norm.

1. Consider the equations,

$$2x_1 + 3x_2 - x_3 = 5, \quad 4x_1 + 4x_2 - 3x_3 = 3, \quad -2x_1 + 3x_2 - x_3 = 1.$$

a) Write the system in the form $(A \,|\, b)$ and solve for $x = (x_1, x_2, x_3)^T$ by Gaussian elimination (i.e. reduction to upper triangular form and back substitution, no pivoting). What are the multipliers? What are the pivots?

b) Find the LU factorization of A and check that $LU = A$. Solve for x by forward and back substitution, i.e. $Ly = b, Ux = y$.

c) Compute the determinant of A two ways, first by the usual method and second by the formula $\det A = a_{11}^{(1)} a_{22}^{(2)} a_{33}^{(3)}$, where $a_{kk}^{(k)}$ is the pivot element in step k of Gaussian elimination.

2. Let $A = \begin{pmatrix} -2 & 1 \\ 2 & 0 \end{pmatrix}$.

a) Find $\dfrac{||Ax||}{||x||}$ for the following three vectors. $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) Find a vector x such that $\dfrac{||Ax||}{||x||} = ||A||$.

3. Let $A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}$, $x = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $b = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix}$.

a) Show that A is invertible and that x is the exact solution of $Ax = b$.

b) Let $\tilde{x}_1 = \begin{pmatrix} 2.1 \\ -2.1 \end{pmatrix}$, $\tilde{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\tilde{x}_3 = \begin{pmatrix} 0.9911 \\ -0.4870 \end{pmatrix}$. Think of $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ as approximations to the exact solution x. For each case find the error norm $||e|| = ||x - \tilde{x}||$ and residual norm $||r|| = ||b - A\tilde{x}||$. Which case has the smallest error norm? Which case has the smallest residual norm? Does a smaller error norm imply a smaller residual norm? Does a smaller residual norm imply a smaller error norm?

c) Find $||A||, ||A^{-1}||, \kappa(A)$.

d) In class we derived the following relation between the error and the residual.

$$\frac{||e||}{||x||} \leq \kappa(A) \frac{||r||}{||b||}$$

Show that the relation is satisfied for the approximate solutions $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ by computing the left hand side and right hand side of the inequality in each case.

announcement

The midterm exam is on Thursday February 28 in class. The exam will cover all the class material up to and including Thursday February 21. A review sheet with sample problems will be distributed before the exam. You may use a calculator to do arithmetic, but to receive full credit you must show all intermediate steps. You may use one sheet of handwritten notes (i.e. one side of one page, 8.5 in $\times$ 11 in).