Math 371 Winter 2013 Homework 5 due: Thursday March 14

announcement : on Feb 19/21 and March 12/14, both sections will meet in 133 Chrysler

This assignment consists of Matlab programming exercises. In class we considered the two-point boundary value problem $-y'' = r(x)$ for $0 \le x \le 1$, with $y(0) = \alpha, y(1) = \beta$. The function $y(x)$ is the temperature profile in a metal rod, $r(x)$ is a distribution of heat sources, and the temperature is given at the ends of the rod. The numerical solution is $w_i \approx y(x_i)$, where the mesh points are $x_i = ih, h = \frac{1}{n+1}, i = 0 : n+1$. We set $w_0 = \alpha, w_{n+1} = \beta$ and the remaining values, $w_h = (w_1, \ldots, w_n)^T$, are determined by the finite-difference equations, $-D_+D_-w_i = r_i$. This leads to a tridiagonal linear system, $A_h w_h = r_h$, which is solved by the tridiagonal LU method (Thomas algorithm) given in class.

A template for the Matlab code is on the back of this sheet. Copy the template into an m-file and fill in the lines to run the code. The code solves the problem for four values of the mesh size, $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, and it plots the exact solution and numerical solution, and produces a table displaying the error. Note that the tridiagonal LU method does not create the full matrix A_h, but instead uses vectors to store the nonzero matrix elements and numerical solution w_h. This saves memory and is important in realistic applications.

For each problem below submit the plots and error table produced by the code. You may copy and paste the table. Explain the trends in each column of the table. To observe the trends, you may compute some smaller values of h. Does the numerical solution converge to the exact solution as $h \to 0$? What is the rate of convergence?

1. The first problem is the one from class, $-y'' = 25 \sin \pi x, y(0) = 0, y(1) = 1$, with exact solution $y(x) = \frac{25}{\pi^2} \sin \pi x + x$. Make sure your code reproduces the results given in class.

2. The second problem is $-y'' = 25 \cos \pi x, y(0) = 0, y(1) = 1$, in which the right hand side has changed. Find the exact solution $y(x)$ in this case and run the code (after changing the appropriate lines). Compare the results for problems (1) and (2). How does the heat source distribution $r(x)$ affect the temperature profile $y(x)$?

3. The third problem is $\mu v'' - \lambda v = \rho_0 g, v(0) = v(W) = 0$, for a function $v(x)$ defined for $0 \le x \le W$. This problem appears on page 401 of the article "Drag of a Flexible Fiber in a 2D Moving Viscous Fluid", by Luoding Zhu and Charles S. Peskin, in Computers & Fluids, vol. 36, (2007) pp. 398-406. The function $v(x)$ represents a fluid velocity profile in which the fluid is flowing vertically downward in a channel under the influence of gravity. The variable x goes across the channel and the walls correspond to $x = 0, W$. The fluid velocity is zero on the channel walls due to the no-slip condition. The parameters are as follows.

$\mu = \rho_0 \nu$: dynamic viscosity coefficient
$\rho_0 = 3 \times 10^{-4}$ g/cm^2 : fluid density
$\nu = 4$ cm^2/s : kinematic viscosity coefficient
$\lambda = \rho_0 g / V_0$: air resistance coefficient
$g = 980$ cm/s^2 : gravitational constant
$V_0 = 200$ cm/s : terminal velocity amplitude
$W = 10$ cm : channel width

The exact solution is $v(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} - V_0$, where r_1, r_2, c_1, c_2 are constants that depend on the numerical parameters above. Find r_1, r_2 by substituting e^{rx} into the differential equation and then find c_1, c_2 by satisfying the boundary conditions (do this by hand and show your work, but you may check your formulas by comparing with the article which is available online). Solve the problem using the finite-difference scheme $\mu D_+D_-w_i - \lambda w_i = \rho_0 g$. Modify the Matlab code from problems 1 and 2 accordingly.

```
function m371bvp1d
% template for numerical solution of a two-point boundary value problem
% -y''=r, y(0)=alpha, y(1)=beta
clear; clf;
alpha = 0; beta = 1;                          % boundary conditions
for icase=1:4
    n = 2^icase-1; h = 1/(n+1);               % h = mesh size
    xe = 0:0.0025:1;                          % fine mesh for plotting exact solution
    ye = ...;                                 % exact solution on fine mesh
% Set up for numerical solution.
for i=1:n
    xh(i) = i*h;                              % mesh points
    yh(i) = ...;                              % exact solution at mesh points
    a(i) = ...; b(i) = ...; c(i) = ...;       % matrix elements
    r(i) = ...;                               % right hand side vector
end
    r(1) = ...;                               % adjust for BC at x=0
    r(n) = ...;                               % adjust for BC at x=1
    wh = LU_371(a,b,c,r);                     % numerical solution
% output
    table(icase,1) = h;
    table(icase,2) = norm(yh-wh,inf);
    table(icase,3) = norm(yh-wh,inf)/h;
    table(icase,4) = norm(yh-wh,inf)/h^2;
    table(icase,5) = norm(yh-wh,inf)/h^3;
    xplot = [0 xh 1]; wplot = [alpha wh beta];
    subplot(2,2,icase); plot(xe,ye,xplot,wplot,'-o');
    string = sprintf('h=1/%d',n+1); title(string)
end
table
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function w = LU_371(a,b,c,r)
% input:  a, b, c, r - matrix elements and right hand side vector
% output:  w - solution of linear system
n = length(r);
%
% Fill in the steps below using the tridiagonal LU method given in class.
%
% find L, U
%
% solve Lz = r
%
% solve Uw = z
%
```