Math 371 Winter 2013 Homework 6 due: Thursday March 21

1. Consider the linear system, $2x_1 + x_2 = 1, x_1 + 2x_2 = -1$, with solution $x_1 = 1, x_2 = -1$.

a) Write Jacobi's method in component form and take three steps starting from initial guess $x_0 = (0,0)^T$. Present the results in a table with the following format.

column 1: k (iteration step)
column 2: $x_1^{(k)}$ (1st component of computed solution vector)
column 3: $x_2^{(k)}$ (2nd component of computed solution vector)
column 4: $||e_k||$ (error norm)
column 5: $||e_k||/||e_{k-1}||$ (ratio of successive error norms)

Find the iteration matrix B_J and compute $||B_J||, \rho(B_J)$. Does the method converge?

b) Repeat for Gauss-Seidel.

c) Repeat for optimal SOR.

Which method converges fastest? Explain.

2. Find the eigenvalues and eigenvectors of the following matrices. Do this by hand, but you may check your answers using Matlab.

a) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ e) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

3. Consider the linear system $Ax = b$, where $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. In class we applied the Gauss-Seidel method with zero vector for the starting guess and showed that the error is given by $e_k = \left(\frac{1}{4}\right)^k \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for $k \geq 1$. We did this using the eigenvalues and eigenvectors of the iteration matrix B_{GS}. Now apply this procedure to Jacobi's method, i.e. consider the error e_k in Jacobi's method and derive an expression for e_k in terms of the eigenvalues and eigenvectors of the iteration matrix B_J. Use this expression to show that $\lim_{k\to\infty} ||e_k|| = 0$.

4. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Use Matlab to plot $\rho(B_\omega)$, the spectral radius of the SOR iteration matrix as a function of the relaxation parameter ω, for $0 \leq \omega \leq 2$. To evaluate $\rho(B_\omega)$, first create the matrix B_ω and then use the commands `eig, abs, max`. Use 1000 points on the interval $0 \leq \omega \leq 2$ for the plot. The plot of $\rho(B_\omega)$ is typical for the class of matrices to which Young's theorem applies (i.e. symmetric, block tridiagonal, positive definite). Indicate the optimal SOR parameter ω_* and the value ω corresponding to the Gauss-Seidel method. If we didn't know the exact value of the optimal SOR parameter ω_*, then based on the graph of $\rho(B_\omega)$, is it better to underestimate or overestimate the value of ω_*? Explain the reason for your answer.

5. Determine which of the following matrices are positive definite. For those that are positive definite, show that $x^TAx > 0$ for all vectors $x \neq 0$. For those that are not positive definite, find a vector $x \neq 0$ such that $x^TAx \leq 0$.

a) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$