Math 371 Winter 2013 Homework 8 due: Tuesday April 9

1. Recall the matrix A_h in the finite-difference solution of the boundary value problem discussed in class (steady state heat conduction). It was stated in class that $\rho(B_J) = \cos\pi h, \rho(B_{GS}) = \cos^2\pi h, \rho(B_{\omega_*}) = \frac{1-\sin\pi h}{1+\sin\pi h}$, where h is the mesh size. Use MATLAB to plot $\rho(B)$ versus h for $0 \le h \le 1/2$ for each method. Include all three methods on the same plot and label each curve. Plot a small symbol on each curve at $h = 1/2, 1/4, 1/8, 1/16$.

a) Explain why the plot shows that all three methods converge for any mesh size $0 < h \le 1/2$.

b) Explain how to tell from the plot which method converges fastest and which converges slowest.

2. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. In class we computed the e-values λ_1, λ_2 of A and the corrresponding orthonormal e-vectors q_1, q_2. Now consider the linear system $Ax = b$ with $b = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$. Compute $x = \lambda_1^{-1}(q_1^T b)q_1 + \lambda_2^{-1}(q_2^T b)q_2$ by evaluating the expression on the right and check that the result is the solution of the linear system. This formula for x is called the <u>spectral method</u>, a method for solving $Ax = b$ using the e-values and e-vectors of A.

3. Write a MATLAB code that applies the power method and inverse power method to find the largest and smallest eigenvalues of A, starting from the given initial guess $v^{(0)}$.

a) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$, $v^{(0)} = (1,0,0)^T$

b) $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, $v^{(0)} = \frac{1}{\sqrt{3}}(1,1,1)^T$

You may check your code by comparing with the results from the example in class. For each method take six steps, $k = 0:5$, and present the results in a table with the following format.

column 1: k - step index

column 2: $\lambda^{(k)}$ - approximate eigenvalue at step k

column 3: $|\lambda^{(k)} - \lambda|$ - error at step k

column 4: $\dfrac{|\lambda^{(k)} - \lambda|}{|\lambda^{(k-1)} - \lambda|}$ - ratio of errors at step k and $k-1$

In your writeup, give the heading for each column. In columns 3 and 4, use $\lambda = \lambda_1$ for the power method and $\lambda = \lambda_3$ for the inverse power method, where the exact eigenvalues of A are $\lambda_1, \lambda_2, \lambda_3$ and they are ordered so that $|\lambda_1| > |\lambda_2| > |\lambda_3|$. You may compute the exact eigenvalues $\lambda_1, \lambda_2, \lambda_3$ using the `eig` command in MATLAB. In the table present at least six decimal digits. Discuss the trends in each column. Do the results agree with the convergence theory presented in class? If not, explain.

c) In class we showed that the convergence factor for the power method is $(\lambda_2/\lambda_1)^2$. Find the convergence factor for the inverse power method and justify your answer as in class.