chapter 5 : polynomial approximation and interpolation

questions

How can we approximate a given function $f(x)$ by a polynomial $p(x)$?

How can we interpolate a set of data values (x_i, f_i) by a polynomial $p(x)$?

recall

A polynomial of degree n has the form $p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$.

note

There are other forms in which a polynomial may be written, as we shall see.

application

$\int_a^b f(x)dx \;\rightarrow\; \int_a^b p_n(x)dx$, ...

Weierstrass Approximation Theorem

Let $f(x)$ be continuous for $a \le x \le b$. Then for any $\epsilon > 0$, there exists a polynomial $p(x)$ such that $\max\limits_{a \le x \le b} |f(x) - p(x)| \le \epsilon$.

pf : Math 451

note

Given $f(x)$, there are many ways to find an approximating polynomial $p(x)$.

Taylor approximation

Given $f(x)$ and a point $x = a$, the Taylor polynomial of degree n about $x = a$ is

$$p_n(x) \;=\; f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2 + \cdots + \tfrac{1}{n!}f^{(n)}(x-a)^n.$$

The Taylor polynomial satisfies the following conditions.

$$p_n(a) = f(a),\; p_n'(a) = f'(a),\; \ldots,\; p_n^{(n)}(a) = f^{(n)}(a)$$

$f(x) \;=\; p_n(x) + r_n(x)$, $r_n(x)$: remainder , error

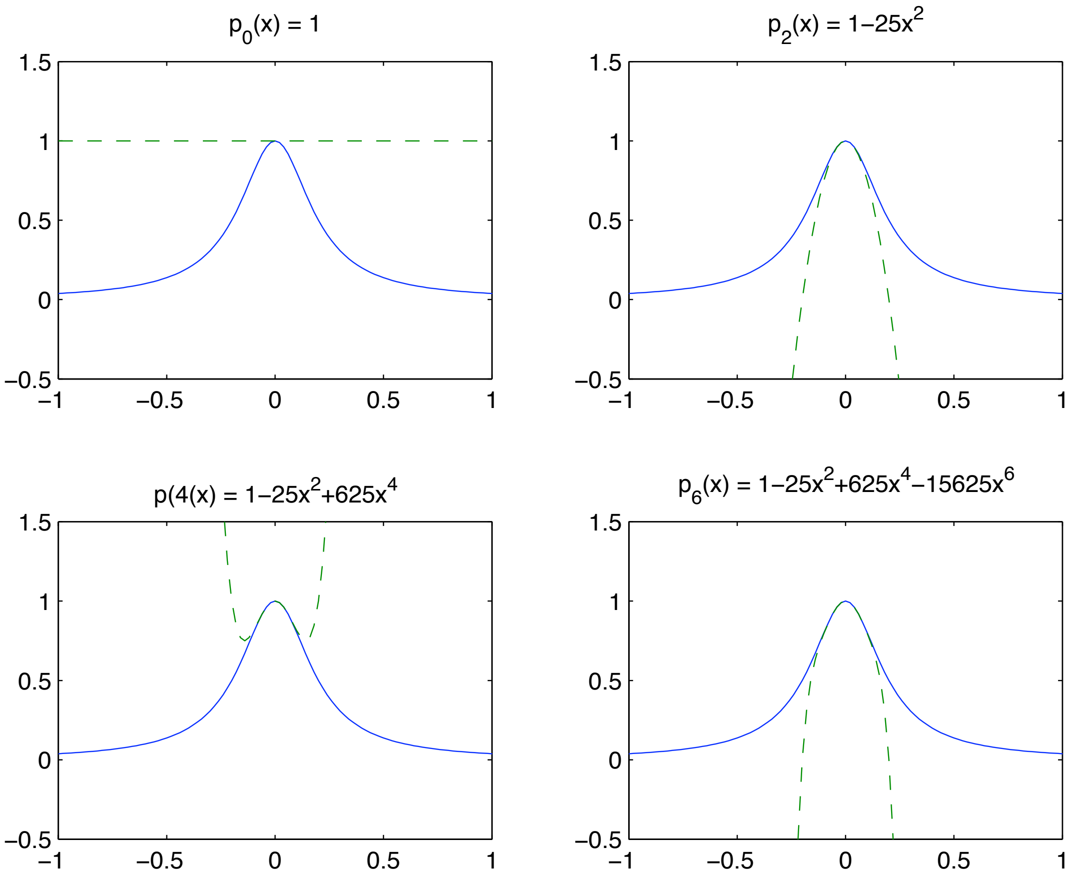
$$r_n(x) = \int_a^x \tfrac{(x-t)^n}{n!} f^{(n+1)}(t)dt = \tfrac{1}{(n+1)!} f^{(n+1)}(\zeta)(x-a)^{n+1} \text{ for some } \zeta \in [a, a+x].$$

$\Rightarrow\; |f(x) - p_n(x)| \le \frac{1}{(n+1)!} \max\limits_{a \le t \le x} |f^{(n+1)}(t)| \cdot |x-a|^{n+1}$: error bound ex : $f(x) = \dfrac{1}{1+25x^2}$, $a = 0$, $p_n(x) = ?$

In this case there is a way to find the Taylor polynomial without computing derivatives, using the formula for the sum of a geometric series.

$$\text{recall}: \ 1 + r + r^2 + r^3 + \cdots = \begin{cases} \dfrac{1}{1-r} & \text{if } |r| < 1 \\ \text{diverges if } |r| \geq 1 \end{cases}$$

$$\frac{1}{1+25x^2} = \frac{1}{1-(-25x^2)} = 1 + (-25x^2) + (-25x^2)^2 + (-25x^2)^3 + \cdots$$



note

$|-25x^2| < 1 \Rightarrow |x| < \frac{1}{5} \Rightarrow \lim_{n\to\infty} p_n(x) = f(x)$, $|x| \geq \frac{1}{5} \Rightarrow \lim_{n\to\infty} p_n(x)$ diverges

explanation : $f(x)$ has complex singularities at $x = \pm\frac{1}{5}i$, Math 555

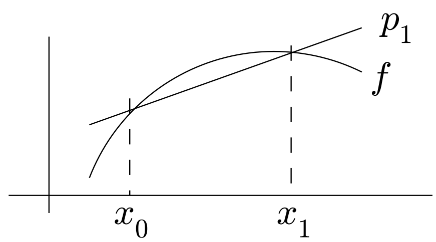
Hence the Taylor polynomial is a good approximation when $f(x)$ is sufficiently differentiable and x is close to a, but in general we need to consider other methods of approximation.

polynomial interpolation

thm: Assume $f(x)$ is given and let $x_0, x_1, \ldots, x_n$ be $n+1$ distinct points. Then there exists a unique polynomial $p_n(x)$, of degree less than or equal to n, which interpolates $f(x)$ at the given points, i.e. such that $p_n(x_i) = f(x_i)$ for $i = 0 : n$.

pf : existence : soon , uniqueness : Fundamental Thm of Algebra, Math 412

ex : $n = 1 \Rightarrow x_0 , x_1$



questions

1. How can we construct $p_n(x)$ for larger n?
2. Is there an efficient way to evaluate $p_n(x)$ for $x \neq x_i$?
3. How large is the error $|f(x) - p_n(x)|$ for $x \neq x_i$?

section 5.1 : Lagrange form of the interpolating polynomial

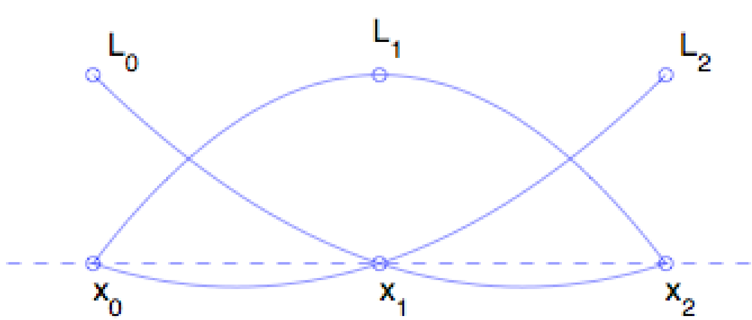
def : $L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \left(\frac{x - x_i}{x_k - x_i} \right)$, $k = 0 : n$: Lagrange polynomial

ex : $n = 2$, $x_0 = -1$, $x_1 = 0$, $x_2 = 1$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{1}{2}x^2 - \frac{1}{2}x$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-(-1))(x-1)}{(0-(-1))(0-1)} = -x^2 + 1$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-(-1))(x-0)}{(1-(-1))(1-0)} = \frac{1}{2}x^2 + \frac{1}{2}x$$



properties

1. $\deg L_k = n$, 2. $L_k(x_i) = \begin{cases} 1 \text{ if } i = k \\ 0 \text{ if } i \neq k \end{cases}$

Now let $f(x)$ be given. The <u>Lagrange form</u> of the interpolating polynomial is

$$p_n(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + \cdots + f(x_n)L_n(x) = \sum_{k=0}^{n} f(x_k)L_k(x).$$

<u>check</u>

1. $\deg p_n \le n$, 2. $p_n(x_i) = \sum_{k=0}^{n} f(x_k)L_k(x_i) = f(x_i)$, $i = 0 : n$ <u>ok</u>

<u>ex</u>

$$f(x) = \frac{1}{1+25x^2}\ ,\ x_0 = -1\,,\ x_1 = 0\,,\ x_2 = 1$$

$$p_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x)$$

$$= \tfrac{1}{26}\cdot(\tfrac{1}{2}x^2 - \tfrac{1}{2}x) + 1\cdot(-x^2+1) + \tfrac{1}{26}\cdot(\tfrac{1}{2}x^2 + \tfrac{1}{2}x) = -\tfrac{25}{26}x^2 + 1$$

check ... <u>ok</u>

<u>note</u>

The Lagrange form is useful theoretically (to show that $p_n(x)$ exists), but it has some practical disadvantages.

1. It is expensive to evaluate $p_n(x)$ for $x \ne x_i$ using this form.
2. Adding an extra point x_{n+1} means that all the $L_k(x)$ must be recomputed.

<u>section 5.3</u> : Newton form of the interpolating polynomial

$$p_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)\cdots(x-x_{n-1})$$

<u>ex</u> : $n = 1$, x_0 , x_1

$$p_1(x) = f(x_0)\left(\frac{x-x_0}{x_0-x_1}\right) + f(x_1)\left(\frac{x-x_1}{x_1-x_0}\right) \ :\ \text{Lagrange form}$$

$$= f(x_0) + \left(\frac{f(x_1)-f(x_0)}{x_1-x_0}\right)(x-x_0) \ :\ \text{Newton form}$$

<u>note</u>

For $n \ge 2$, we need a method to compute $a_0, \ldots, a_n$. Assume that $p_{n-1}(x)$ is already known. Then define $p_n(x) = p_{n-1}(x) + a_n(x-x_0)\cdots(x-x_{n-1})$. We will derive a formula for a_n.

$$i = 0 : n-1 \ \Rightarrow\ p_n(x_i) = p_{n-1}(x_i) + a_n(x_i-x_0)\cdots(x_i-x_{n-1}) = f(x_i)$$

$$i = n \ \Rightarrow\ p_n(x_n) = p_{n-1}(x_n) + a_n(x_n-x_0)\cdots(x_n-x_{n-1}) = f(x_n)$$

$$\Rightarrow\ a_n = \frac{f(x_n) - p_{n-1}(x_n)}{(x_n-x_0)\cdots(x_n-x_{n-1})}$$: This formula is correct, but there is a more efficient way to compute a_n.

First define some notation.

$a_n = f[x_0,\dots,x_n] \;\Rightarrow\; a_0 = f[x_0]\ ,\ a_1 = f[x_0,x_1]\ ,\ a_2 = f[x_0,x_1,x_2]\ ,\ \dots$

claim : $f[x_0,\dots,x_n] = \dfrac{f[x_1,\dots,x_n] - f[x_0,\dots,x_{n-1}]}{x_n - x_0}$: divided difference

pf

$p_{n-1}(x)$ interpolates $f(x)$ at $x_0,\dots,x_{n-1}$, $\deg p_{n-1} \le n-1$

$q_{n-1}(x)$ interpolates $f(x)$ at $x_1,\dots,x_n$, $\deg q_{n-1} \le n-1$

define $g(x) = \left(\dfrac{x-x_0}{x_n-x_0}\right) q_{n-1}(x) + \left(\dfrac{x_n-x}{x_n-x_0}\right) p_{n-1}(x)$

then $\deg g \le n$

$g(x_0) = p_{n-1}(x_0) = f(x_0)$

$g(x_n) = q_{n-1}(x_n) = f(x_n)$

$i = 1 : n-1 \;\Rightarrow\; g(x_i) = \left(\dfrac{x_i-x_0}{x_n-x_0}\right) q_{n-1}(x_i) + \left(\dfrac{x_n-x_i}{x_n-x_0}\right) p_{n-1}(x_i) = f(x_i)$

$\Rightarrow\; g(x) = p_n(x)$, now equate the coefficients of x^n in these two polynomials

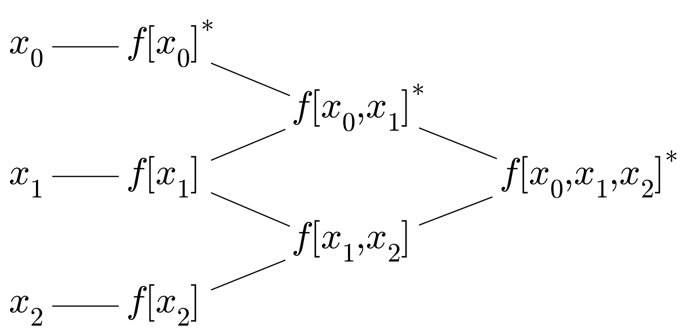
$\Rightarrow\; \dfrac{f[x_1,\dots,x_n]}{x_n-x_0} - \dfrac{f[x_0,\dots,x_{n-1}]}{x_n-x_0} = f[x_0,\dots,x_n]$ ok

Hence we can rewrite Newton's form of the interpolating polynomial as follows.

$$p_n(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \cdots + f[x_0,\dots,x_n](x-x_0)\cdots(x-x_{n-1})$$

The coefficients can be efficiently computed using a divided difference table.

ex : $n = 2$



The starred values are the coefficients in Newton's form of $p_2(x)$.

$p_2(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$

ex: $f(x) = \frac{1}{1+25x^2}$, $x_0 = -1$, $x_1 = 0$, $x_2 = 1$

$$\begin{array}{cccc} -1 & \frac{1}{26} & & \\ & & \frac{25}{26} & \\ 0 & 1 & & -\frac{25}{26} \\ & & -\frac{25}{26} & \\ 1 & \frac{1}{26} & & \end{array} \Rightarrow \quad \begin{aligned} p_2(x) &= \tfrac{1}{26} + \tfrac{25}{26}(x-(-1)) - \tfrac{25}{26}(x-(-1))(x-0) \\ &= 1 - \tfrac{25}{26}x^2 \qquad \underline{\text{ok}} \end{aligned}$$

note

Using Newton's form for $p_n(x)$, if an extra interpolation point x_{n+1} is added, then the previous terms remain unchanged.

evaluation of $p_n(x)$

$p_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$: Newton form , 3 mults

$\quad = a_0 + (x - x_0)(a_1 + a_2(x - x_1))$: nested form , 2 mults

general case

$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1})$

$\quad = a_0 + (x - x_0)(a_1 + (x - x_1)(a_2 + \cdots + a_n(x - x_{n-1}))\cdots)$

operation count : Newton form $= \frac{n(n+1)}{2}$ mults , nested form $= n$ mults

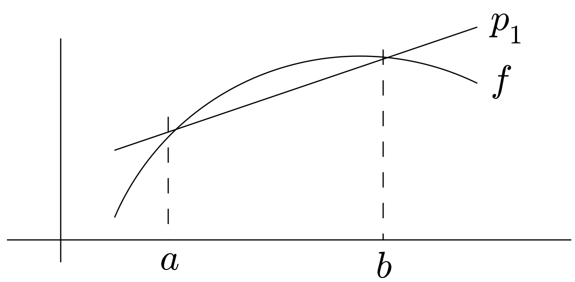
thm (the error in polynomial interpolation)

$f(x)$, $x_0 < \ldots < x_n$: given , $p_n(x)$: interpolating polynomial

Then $x_0 \le x \le x_n \Rightarrow f(x) = p_n(x) + \dfrac{1}{(n+1)!} f^{(n+1)}(\zeta)(x - x_0)\cdots(x - x_n)$.

pf : omit , but note that this resembles the error in Taylor approximation

application : $n = 1$, $x_0 = a$, $x_1 = b$



$a \le x \le b \Rightarrow |f(x) - p_1(x)| \le \frac{1}{8}M|b - a|^2$, where $M = \max_{a \le x \le b} |f''(x)|$

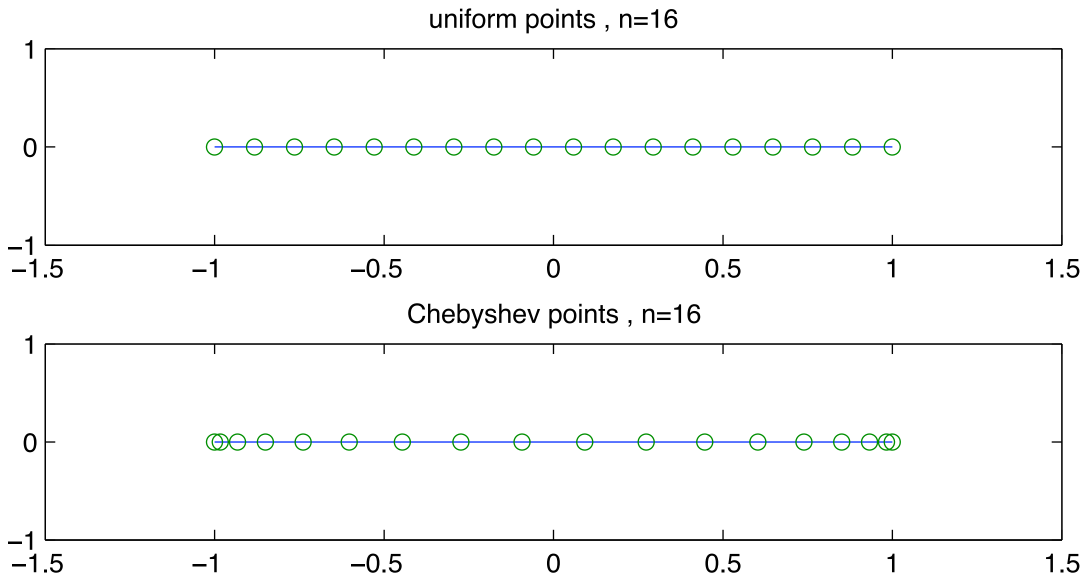
pf : hw8

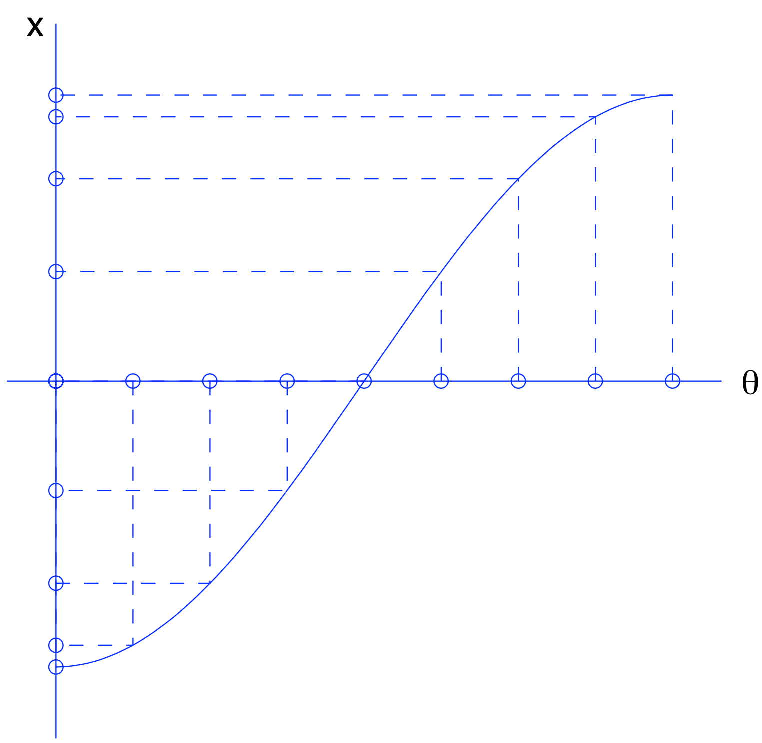
section 5.4 : optimal points for interpolation

Given $f(x)$ for $-1 \le x \le 1$, how should we choose the interpolation points $x_0, \dots, x_n$? Consider two options.

1. uniform points : $x_i = -1 + ih$, $h = \frac{2}{n}$, $i = 0 : n$

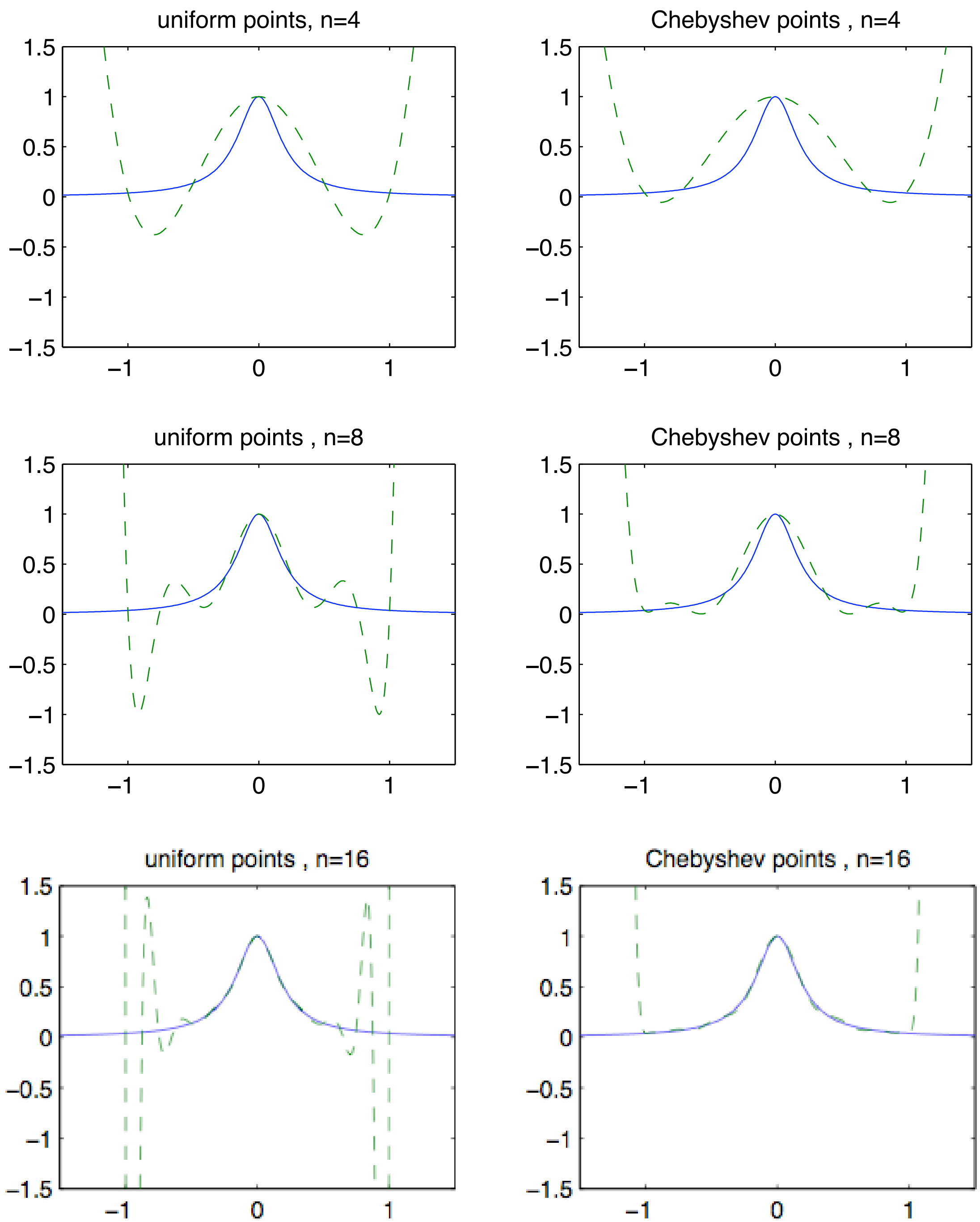
2. Chebyshev points : $x_i = -\cos\theta_i$, $\theta_i = i\frac{\pi}{n}$, $i = 0 : n$





note : The Chebyshev points are clustered near the endpoints of the interval.

$\underline{\text{ex}}$: $f(x) = \dfrac{1}{1+25x^2}$, polynomial interpolation on the interval $-1 \le x \le 1$

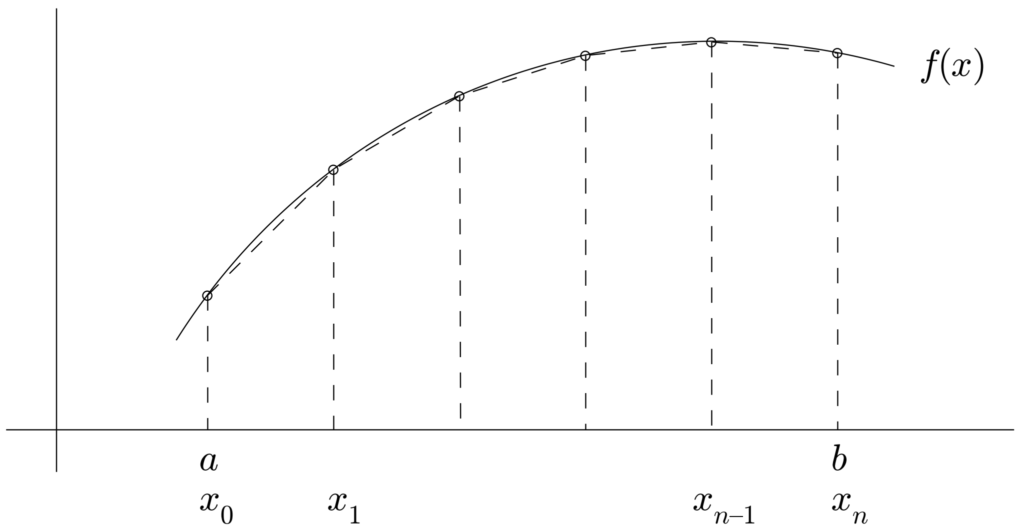


1. Interpolation at the uniform points gives a good approximation near the center of the interval, but it gives a bad approximation near the endpoints.

2. Interpolation at the Chebyshev points gives a good approximation on the entire interval.

section 5.5 : piecewise linear interpolation

Given $f(x)$, $a \le x \le b$, $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.



The interpolating polynomial $p_n(x)$ may not be a good approximation to $f(x)$ on the entire interval, so instead we may consider the piecewise linear interpolant, denoted by $q(x)$.

$x_i \le x \le x_{i+1} \Rightarrow q(x) = f[x_i] + f[x_i, x_{i+1}](x - x_i) \Rightarrow q(x_i) = f(x_i)$, $i = 0 : n$

error : $|f(x) - q(x)| \le \frac{1}{8} \max_{a \le x \le b} |f''(x)| \cdot \max_i |x_{i+1} - x_i|^2$: 2nd order accurate

note : $q(x)$ is continuous, but it is not differentiable at $x = x_i$.

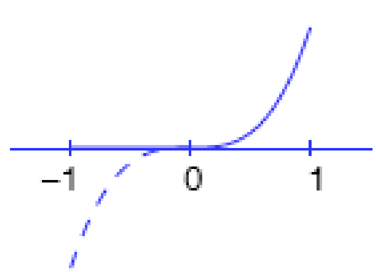
section 5.6 : piecewise cubic interpolation

A cubic spline is a function $s(x)$ satisfying the following conditions.

1. For each interval $x_i \le x \le x_{i+1}$, $s(x)$ is a cubic polynomial.
2. $s(x)$, $s'(x)$, $s''(x)$ are continuous at the interior points $x_1, \ldots, x_{n-1}$

ex : $x_0 = -1$, $x_1 = 0$, $x_2 = 1$

$$s(x) = \begin{cases} 0 & , -1 \le x \le 0 \\ x^3 & , \quad 0 \le x \le 1 \end{cases}$$



$$s'(x) = \begin{cases} 0 & , -1 \le x \le 0 \\ 3x^2 & , \quad 0 \le x \le 1 \end{cases}$$

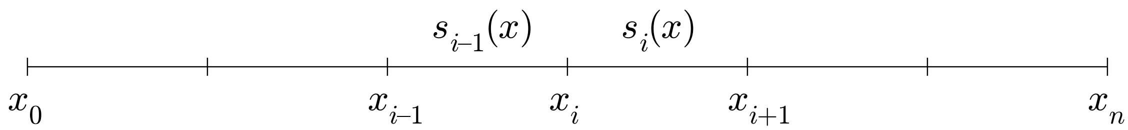
$$s''(x) = \begin{cases} 0 & , -1 \le x \le 0 \\ 6x & , \quad 0 \le x \le 1 \end{cases}$$

Hence $s(x)$ satisfies the conditions required to be a cubic spline.

<u>problem</u>

Given $f(x)$ and $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Find a cubic spline $s(x)$ which interpolates $f(x)$ at the given points, i.e. $s(x_i) = f(x_i)\,,\ i = 0 : n$.

$x_i \le x \le x_{i+1} \ \Rightarrow\ s(x) = s_i(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3\ ,\ i = 0 : n-1$



$n+1$ points $\Rightarrow$ n intervals $\Rightarrow$ $4n$ unknown coefficients

interpolation conditions $\Rightarrow$ $2n$ equations

continuity of $s'(x)\,,\ s''(x)$ at interior points $\Rightarrow$ $2(n-1)$ equations

We can choose 2 more equations. A popular choice is to set $s''(x_0) = s''(x_n) = 0$, which gives the <u>natural cubic spline interpolant</u>.

<u>how to find $s(x)$</u>

<u>ex</u> : $-1 \le x \le 1\ ,\ x_i = -1 + ih\ ,\ h = \frac{2}{n}\ ,\ i = 0, \ldots, n$: uniform points

<u>step 1</u> : 2nd derivative conditions

$s_i''(x)$ is a linear polynomial $\Rightarrow$ $s_i''(x) = a_i\left(\dfrac{x_{i+1} - x}{h}\right) + a_{i+1}\left(\dfrac{x - x_i}{h}\right)$

$\Rightarrow\ s_i''(x_i) = a_i\ ,\ s_i''(x_{i+1}) = a_{i+1}\ \Rightarrow\ s_{i-1}''(x_i) = a_i = s_i''(x_i)$

Hence $s''(x)$ is continuous at the interior points.

<u>step 2</u> : interpolation

integrate twice

$$s_i(x) = \frac{a_i(x_{i+1} - x)^3}{6h} + \frac{a_{i+1}(x - x_i)^3}{6h} + b_i\left(\frac{x_{i+1} - x}{h}\right) + c_i\left(\frac{x - x_i}{h}\right)$$

$$s_i(x_i) = \frac{a_i h^2}{6} + b_i = f_i \ \Rightarrow\ b_i = f_i - \frac{a_i h^2}{6}$$

$$s_i(x_{i+1}) = \frac{a_{i+1} h^2}{6} + c_i = f_{i+1} \ \Rightarrow\ c_i = f_{i+1} - \frac{a_{i+1} h^2}{6}$$

$$\begin{aligned} s_i(x) = {} & \frac{a_i(x_{i+1} - x)^3}{6h} + \frac{a_{i+1}(x - x_i)^3}{6h} \\ & + \left(f_i - \frac{a_i h^2}{6}\right)\left(\frac{x_{i+1} - x}{h}\right) + \left(f_{i+1} - \frac{a_{i+1} h^2}{6}\right)\left(\frac{x - x_i}{h}\right) \end{aligned}$$

step 3 : 1st derivative conditions

$$s_i'(x) = -\frac{a_i(x_{i+1}-x)^2}{2h} + \frac{a_{i+1}(x-x_i)^2}{2h}$$
$$+\left(f_i - \frac{a_i h^2}{6}\right)\cdot\frac{-1}{h} + \left(f_{i+1} - \frac{a_{i+1}h^2}{6}\right)\cdot\frac{1}{h}$$

$$s_i'(x_i) = -\frac{a_i h}{2} - \frac{f_i}{h} + \frac{a_i h}{6} + \frac{f_{i+1}}{h} - \frac{a_{i+1}h}{6}$$

$$s_i'(x_{i+1}) = \frac{a_{i+1}h}{2} - \frac{f_i}{h} + \frac{a_i h}{6} + \frac{f_{i+1}}{h} - \frac{a_{i+1}h}{6}$$

we require $s_{i-1}'(x_i) = s_i'(x_i)$

$$\frac{a_i h}{2} - \frac{f_{i-1}}{h} + \frac{a_{i-1}h}{6} + \frac{f_i}{h} - \frac{a_i h}{6} = -\frac{a_i h}{2} - \frac{f_i}{h} + \frac{a_i h}{6} + \frac{f_{i+1}}{h} - \frac{a_{i+1}h}{6}$$

$$\frac{a_{i-1}h}{6} + a_i\left(\frac{h}{2} - \frac{h}{6} + \frac{h}{2} - \frac{h}{6}\right) + \frac{a_{i+1}h}{6} = \frac{f_{i-1} - 2f_i + f_{i+1}}{h}$$

$$a_{i-1} + 4a_i + a_{i+1} = \frac{6}{h^2}\left(f_{i-1} - 2f_i + f_{i+1}\right)\,,\ i = 1:n-1$$

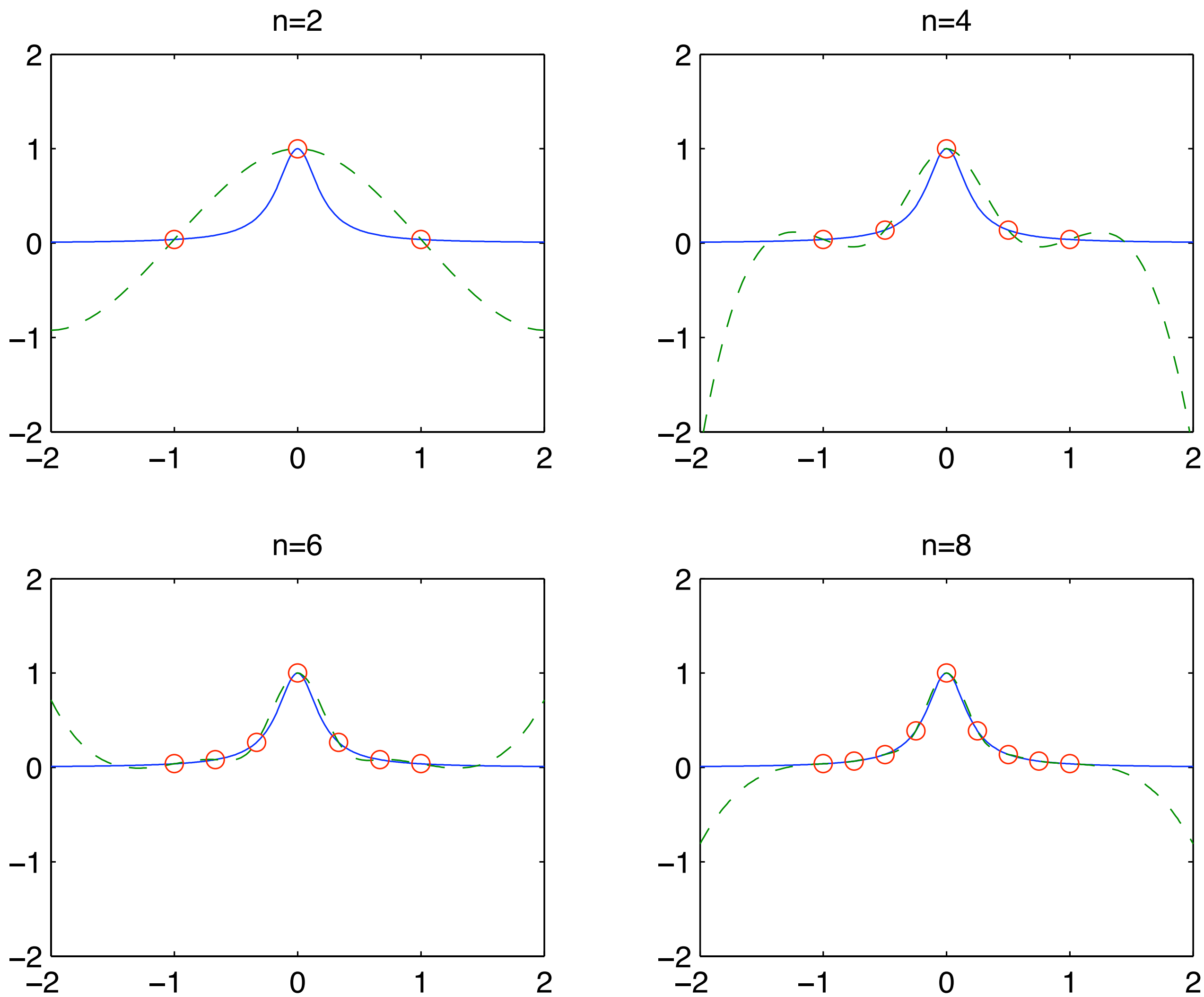
step 4 : BC

$s_0''(x_0) = 0 \;\Rightarrow\; a_0 = 0\,,\ s_{n-1}''(x_n) = 0 \;\Rightarrow\; a_n = 0$

$$\begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ \vdots \\ \vdots \\ a_{n-1} \end{pmatrix} = \frac{6}{h^2}\begin{pmatrix} f_0 - 2f_1 + f_2 \\ \vdots \\ \vdots \\ \vdots \\ f_{n-2} - 2f_{n-1} + f_n \end{pmatrix}$$

A : tridiagonal , symmetric , positive definite ex : natural cubic spline interpolation

$$f(x) = \frac{1}{1+25x^2} \ , \quad -1 \le x \le 1 \ , \ x_i = -1 + ih \ , \ h = \frac{2}{n} \ , \ i = 0 : n$$



note

1. $|f(x) - s(x)| \le \dfrac{5}{384} \max\limits_{a \le x \le b} |f^{(4)}(x)|\, h^4$: 4th order accurate , pf : omit

2. The natural cubic spline interpolant has inflection points at the endpoints of the interval, due to the boundary conditions $s''(x_0) = s''(x_n) = 0$. In fact, there are additional inflection points in the interior of the interval, which are problematic in some applications.