Math 471   Review Sheet for Final Exam   Fall 2009

The final exam will cover the entire course. You may use two sheets of notes (e.g. two sides of one page, i.e. a total of $187\,\text{in}^2 = 2 \times 8.5\,\text{in} \times 11\,\text{in}$). You may use a non-programmable calculator to do arithmetic, but to receive full credit you must show all intermediate steps. Exam booklets will be provided. Please note the following schedule.
review class (4088 EH): Wed 12/16, 11am-12pm
office hours (4830 EH): Fri 12/18, 3-4pm; Mon 12/21, 3-4pm; Tues 12/22, 12-1pm
final exam (1084 EH): Wed 12/23, 10:30am-12:30pm

1. True or False? Give a reason to justify your answer.

a) If two floating point numbers with n significant digits are added, then the result also has n significant digits.

b) $D_+D_-y_i = D_-D_+y_i$ c) $D_0f(x) = f'(x) + O(h^2)$ d) If $Ax = 0$, then $A = 0$ or $x = 0$.

e) If A is invertible, then $||A||^{-1} \le ||A^{-1}||$.   f) $\rho(B) \le ||B||_\infty$ for any matrix B

g) The spectral radius of a matrix satisfies the properties required to be a matrix norm.

h) If the pivots arising in Gaussian elimination are nonzero, then A is invertible.

i) In solving an $n \times n$ system of linear equations by Gaussian elimination, if n increases by a factor of 5, then the operation count increases by a factor of approximately 25.

j) In computing the solution of a linear system $Ax = b$, if the residual norm $||r||$ is small, then the error norm $||e||$ is also small.

k) In solving a linear system $Ax = b$ by an iterative method $x_{k+1} = Bx_k + c$, if $||B||_\infty < 1$, then $\lim_{k \to \infty} x_k = x$ for any initial guess x_0.

l) In solving a 2-point boundary value problem using a finite-difference scheme with mesh size h, if Jacobi's method is used to solve the linear system, then the spectral radius of the iteration matrix satisfies $\lim_{h \to 0} \rho(B_J) = 0$.

m) Suppose a two-dimensional boundary value problem is solved using a finite-difference scheme and the resulting linear system is solved by Jacobi's method with stopping criterion $||r_k||_\infty \le 10^{-2}$. If the mesh size h is decreased, then the number of iterations needed to satisfy the stopping criterion is increased.

n) The method called "shifted inverse iteration" is used to find the inverse of a matrix.

o) Wilkinson's example shows that the coefficients of a polynomial can depend sensitively on the roots.

p) When the power method is applied to find the largest eigenvalue and the corresponding eigenvector of a matrix, the vectors are normalized at each step in order to increase the rate of convergence of the method.

q) If $p_n(x)$ is the interpolating polynomial of degree n for a given function $f(x)$ at points $x_i = a + ih$, where $h = \frac{b-a}{n}$ and $i = 0 : n$, then $\lim_{n \to \infty} p_n(x) = f(x)$ for all $x \in [a, b]$.

r) If $p_n(x)$ is the Taylor polynomial of degree n for $f(x)$ about $x = a$, then $p_n^{(n+1)}(a) = 0$.

s) If x_0, x_1, x_2 are three distinct points and $p(x) = L_0(x) + L_1(x) + L_2(x)$, then $p'(x_i) = 0$ for $i = 1, 2, 3$.

t) Chebyshev points are advantageous for polynomial interpolation because they are clustered near the center of the interval.

u) Suppose $f(x)$ is approximated by a cubic spline interpolant $s(x)$ on the interval $a \le x \le b$ with interpolation points $x_i = a + ih$, where $h = \frac{b-a}{n}$ and $i = 0 : n$. Then if n is doubled, the error defined by $\max_{a \le x \le b} |f(x) - s(x)|$ is reduced by a factor of approximately $1/16$.

2. State one advantage of ...
a) ... Newton's method over the secant method.
b) ... Gaussian elimination with pivoting over Gaussian elimination without pivoting.
c) ... optimal SOR over Gauss-Seidel in solving.
d) ... Rayleigh quotient iteration over the power method.
e) ... polynomial interpolation over Taylor approximation.
f) ... Newton's form for the interpolating polynomial over Lagrange's form.
g) ... cubic spline interpolation over piecewise linear interpolation.

chapter 1, finite-difference approximations

3. Consider the following approximation for the first derivative, $f'(x) \approx \frac{-3f(x)+4f(x+h)-f(x+2h)}{2h}$. The discretization error has the form, error $= c \cdot \max |f^{(m)}(\zeta)| \cdot h^n$. Find the constants c, m, n.

4. Let $f(x) = e^x$ and let $D(h)$ denote the forward difference approximation of the first derivative, $f'(x) \approx D_+ f(x)$ with $h = 1, \frac{1}{2}, \frac{1}{4}$. Apply Richardson extrapolation to obtain more accurate values.

chapter 2, rootfinding

5. Consider solving the equation $f(x) = x^2 - 5 = 0$ by the bisection method.
a) Explain why $0 \le x \le 4$ is a suitable starting interval for this method.
b) Take 3 steps of the bisection method, i.e. compute x_0, x_1, x_2.
c) Approximately how many steps are needed to ensure that the error is less than 10^{-4}?

6. Suppose fixed-point iteration is applied to the function $g(x) = x^2 - \frac{1}{2}x + \frac{1}{2}$. Find the fixed points and in each case determine whether the iteration converges for starting values sufficiently close to the fixed point.

chapter 3, numerical linear algebra, BVP

7. Which of the following matrices are positive definite? a) $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, b) $\begin{pmatrix} 4 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix}$

8. Let $A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}$.
a) Find a vector x such that $||Ax||_\infty = ||A||_\infty$.
b) Find an approximate e-value of A by taking one step of the power method with initial guess $v^{(0)} = \frac{1}{\sqrt{3}}(1, 1, 1)^T$.
c) If $Ax = b$ is solved by Jacobi's method, does the iteration converge for any initial guess?
d) Answer the same question for Gauss-Seidel.
e) Explain how to find the optimal SOR parameter ω_*.

9. Prove the following results.
a) If A is positive definite, then A is invertible.
b) If A is positive definite, then the diagonal elements are positive.
c) If A is positive definite, then the eigenvalues are positive.
d) If A is invertible, then $A^T A$ is symmetric and positive definite.

10. The SOR method in component form for a 2×2 system is given below. Find and correct any errors.
$$a_{11}x_1^{(k+1)} = a_{11}x_1^{(k)} - \omega\left(a_{11}x_1^{(k)} + a_{12}x_2^{(k)} - b_1\right)$$
$$a_{22}x_2^{(k+1)} = a_{22}x_2^{(k)} - \omega\left(a_{21}x_1^{(k+1)} + a_{22}x_2^{(k+1)} - b_2\right)$$

11. The two-point boundary value problem $y'' - y = x, y(0) = 1, y(1) = 0$ for $0 \le x \le 1$ is solved by the finite-difference scheme $D_+D_-w_i - w_i = 0$ for $i = 1 : n$ with step size $h = 1/(n+1)$ and $w_0 = 1, w_{n+1} = 0$. Using $n = 3$, write down the linear system $A_hw_h = f_h$.

12. Consider the Poisson equation $-\Delta\phi = f$ with boundary condition $\phi = g$, on the unit square $0 \le x, y \le 1$. Let the domain be discretized with mesh size $h = \frac{1}{4}$. Then there are nine unknown values in the interior of the domain, w_{ij}, for $i, j = 1, 2, 3$. Suppose the equation is discretized using the finite-difference scheme discussed in class and the linear system is written as $A_hw_h = f_h$, where the elements of w_h have the ordering shown in the figure (this is called the red-black ordering and it's different than the ordering considered in class). Write down the matrix A_h in this case.
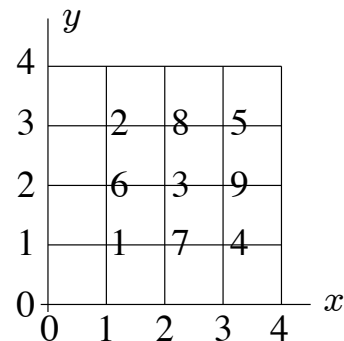
13. Let $A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$. How many steps of Jacobi's method are required to reduce the error as much as one step of the Gauss-Seidel method?

14. The formulas $\rho(B_J) = \cos \pi h, \rho(B_{GS}) = \cos^2 \pi h, \rho(B_{\omega_*}) = \frac{1-\sin \pi h}{1+\sin \pi h}$ were given in class for the spectral radius of the iteration matrix of a finite-difference scheme applied to a boundary value problem. Graph these formulas as a function of πh on the same plot for $0 \le \pi h \le \pi$. Label each formula. What do the graphs imply about the convergence of the methods?

15. Apply the spectral method to solve $Ax = b$, where $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, b = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

chapter 4, computing eigenvalues

16. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Find $\max_{x \ne 0} R_A(x)$.

17. Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Take two steps of the power method starting from $v^{(0)} = \frac{1}{\sqrt{3}}(1, 1, 1)^T$, i.e. compute $v^{(k)}, R_A(v^{(k)})$ for $k = 1, 2$.

chapter 5, polynomial approximation and interpolation

18. Let $p(x) = 1 + x + x^2$ and take $x_0 = -1, x_1 = 0, x_2 = 1$. Write $p(x)$ in (a) Lagrange form, (b) Newton form, (c) nested form.

19. The outdoor temperature $T(t)$ is recorded at two-hour intervals starting at 8am and ending at 4pm, but the 12pm measurement was accidentally omitted. The recorded temperatures are $T(8am) = 30°$ F, $T(10am) = 40°$ F, $T(2pm) = 50°$ F, $T(4pm) = 60°$ F. Use a cubic interpolating polynomial to estimate the missing temperature.

20. Let $f(x)$ be a given function and let $T_1(x)$ denote the Taylor polynomial of degree 1 for $f(x)$ about a given point $x = a$. Show that $f(x) = T_1(x) + \int_a^x (x - t)f''(t)dt$. (This gives the integral form for the error in linear Taylor approximation.)

21. Find the natural cubic spline $s(x)$ satisfying $s(0) = 0, s(1/2) = 1, s(1) = 0$. Your answer will be 2 cubic polynomials, $s_0(x)$ defined on the interval $0 \leq x \leq \frac{1}{2}$ and $s_1(x)$ defined on the interval $\frac{1}{2} \leq x \leq 1$.

chapter 6, numerical integration

22. In class we computed the integral $\int_0^1 e^{-x^2} dx$ using the trapezoid rule and we saw that the method converges with 2nd order accuracy. Now apply the trapezoid rule to compute $\int_0^1 \sqrt{x} dx$. Present the results in a table with the following columns. column 1: h (take $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$), column 2: $T(h)$, column 3: $|I - T(h)|$, column 5: $|I - T(h)|/h^2$, where I is the exact value. What can you say about the order of accuracy in this case? Explain.

23. Consider the integral $I = \int_0^1 x e^{-x^2} dx$.
a) Compute I using the trapezoid rule and Richardson extrapolation with $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.
b) Estimate how small h must be for the trapezoid rule to give the same error as in the last column of the table.

24. In class it was stated that the error in the trapezoid rule has an asymptotic expansion in the form $T(h) = \int_a^b f(x) dx + c_2 h^2 + c_4 h^4 + c_6 h^6 \cdots$, where the c_i are constants. Find c_2.

25. Consider an integration rule of the form $\int_0^{2h} f(x) dx \approx c_0 f(0) + c_1 f(h) + c_2 f(2h)$.
a) Find values of the constants which ensure the rule is exact for polynomials of degree 0,1,2.
b) Find the order of accuracy of the resulting scheme. (Note: this is called <u>Simpson's rule</u>.)

26. The local form of the <u>midpoint rule</u> is $\int_0^h f(x) dx \sim c f(\frac{1}{2}h)$.
a) Find the value of the constant c which ensures that the midpoint rule is exact for constant functions. Show that the resulting method is also exact for linear functions.
b) Is the midpoint rule more accurate or less accurate than the trapezoid rule?

27. The Legendre polynomials $P_i(x), i = 0 : 3$ were derived in class.
a) Apply the Gram-Schmidt process to find $P_4(x)$.
b) Express x^4 as a linear combination of $P_i(x), i = 0 : 4$.

28. Evaluate the integral $\int_0^{2\pi} e^{-x} \sin x \, dx$ using the following methods.
a) trapezoid rule with $h = 2\pi, \pi, \frac{1}{2}\pi$
b) Richardson extrapolation applied to the results in (a)
c) three-point Gaussian quadrature
d) the methods of Calculus I (e.g. integration by parts, FTC, etc.)

29. Among the functions $\{1, x, x^2, \sin \pi x, \cos \pi x, \sin^2 \pi x\}$, find all pairs that are orthogonal with respect to the inner product $\langle f, g \rangle = \int_{-1}^1 f(x) g(x) dx$.

30. A general inner product of two functions defined on the interval $[-1, 1]$ is given by the formula $\langle f, g \rangle = \int_{-1}^1 f(x) g(x) w(x) dx$, where $w(x) \geq 0$ is a given weight function. In class we considered the case $w(x) = 1$ and we applied the Gram-Schmidt process to the set $\{1, x, x^2, \ldots\}$ to obtain the Legendre polynomials. The Chebyshev polynomials $T_n(x)$ are defined similarly, but with respect to the weight function $w(x) = (1 - x^2)^{-1/2}$.
a) Find the Chebyshev polynomials $T_n(x)$ for $n = 0, 1, 2, 3$.
b) In the same way as we derived the Gaussian quadrature rule, this leads to the Gauss-Chebyshev quadrature rule. For example, the 3-point Gauss-Chebyshev rule has the form $\int_{-1}^1 f(x)(1 - x^2)^{-1/2} dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$. Find $c_1, c_2, c_3, x_1, x_2, x_3$ and verify that the rule is exact when $f(x)$ is a polynomial of degree ≤ 5.