Math 471 Fall 2009 Homework 5 due: Fri Nov 13

1. Find the eigenvalues and eigenvectors of the following matrices. Do this by hand, but you may check your answers using Matlab.

a) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ d) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ e) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

section 3.7, special matrices

2. Show that the following matrices are positive definite.

a) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

section 3.8, iterative methods

3. Consider $Ax = b$, where $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

a) In class we showed that the error in the Gauss-Seidel method is given by

$$e_k = \left(\frac{1}{4}\right)^k \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

for $k \geq 1$, using the eigenvalues and eigenvectors of the iteration matrix B_{GS}. Following the same procedure, derive an analogous expression for the error in Jacobi's method. Assume that the starting guess is the zero vector.

b) Use Matlab to plot $\rho(B_w)$, the spectral radius of the SOR iteration matrix, for $0 \leq \omega \leq 2$. Use the commands `eig, abs, max` to produce the plot. Make sure to use a fine enough mesh in the variable ω to resolve the details of the function (1000 points on the interval $0 \leq \omega \leq 2$ is sufficient.) This plot is typical for the type of matrices appearing in Young's theorem. Suppose we don't know the exact value of the optimal SOR parameter - in using an approximate value for the iterative method, is it better to overestimate or underestimate the value of ω_*? Explain the reason behind your answer.

4. Consider the iteration $x_{k+1} = Bx_k + c$ and assume that $||B|| = \alpha < 1$. We know that the error satisfies $||x - x_{k+1}|| \leq \alpha ||x - x_k||$, which is an important theoretical error bound, but the right side cannot be computed in practice because although we know x_k, we don't know x. Here we derive an alternative error bound that can be computed in practice.

a) Show that $I - B$ is invertible and that $||(I - B)^{-1}|| \leq \frac{1}{1-\alpha}$.

b) Show that $||x - x_{k+1}|| \leq \frac{\alpha}{1-\alpha} ||x_{k+1} - x_k||$.

Note: this bound can be computed because we know x_{k+1} and x_k.