Math 471 Fall 2009 Homework 6 due: Wed Nov 25

chapter 4, computing eigenvalues

1. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and recall that in class we computed the e-values λ_1, λ_2 and corresponding orthonormal e-vectors q_1, q_2 of A. Now consider the linear system $Ax = b$, where $b = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$. Compute $x = \lambda_1^{-1}(q_1^T b)q_1 + \lambda_2^{-1}(q_2^T b)q_2$ by explicitly evaluating the expression on the right and check that the result is the solution of the linear system. This illustrates the spectral method, a method that expresses the solution of $Ax = b$ in terms of the e-values and e-vectors of A.

2. In class we considered the e-value problem for the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$. The largest e-value is $\lambda_1 = 5.214319743377535$ (as can be confirmed using the Matlab `eig` command). A table of results was presented for the power method, shifted inverse iteration with $\mu = 5$, and Rayleigh quotient iteration. The initial guess for the e-vector was $v^{(0)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. The computed e-values $\lambda^{(k)}$ were given for iterations $k = 0, 1, 2$. Repeat the calculations and extend the table to $k = 3$. Use Matlab and present 15 decimal digits for the computed e-values (type `format long`). For each method, underline the correct digits at the final step $k = 3$.

Computing Project 2 , due: Wed Nov 25

chapter 9, two-dimensional boundary value problems

A metal plate is described by the unit square $D = \{(x, y) : 0 \le x, y \le 1\}$. The temperature $\phi(x, y)$ satisfies the Laplace equation $\phi_{xx} + \phi_{yy} = 0$ on D with boundary conditions $\phi(x, 1) = 1, \phi(x, 0) = \phi(0, y) = \phi(1, y) = 0$. Physically this means that there are no heat sources in the plate, and one side of the plate is heated to a constant temperature while the other three sides are kept at a lower temperature. Solve for the temperature using the finite-difference scheme $(D_+^x D_-^x + D_+^y D_-^y)w_{ij} = 0$ with mesh size $h = \frac{1}{n+1}$. This yields a linear system denoted by $A_h w_h = f_h$, where $w_h = \{w_{ij}\}$ is the numerical solution vector with components $w_{ij} \approx \phi(x_i, y_j)$ and the mesh points are given by (x_i, y_j) with $x_i = ih, y_j = jh, i, j = 0 : n+1$. Use Matlab to solve the linear system by Jacobi's method with mesh sizes $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$. Let $w_h^{(0)} = 0$ be the starting vector. Take $||r_k||_\infty / ||r_0||_\infty \le 10^{-2}$ for the stopping criterion, where $r_k = A_h w_h^{(k)} - f_h$ is the residual at step k. To keep the code simple, the numerical solution $w_h = \{w_{ij}\}$ should be coded as a matrix of dimension $(n + 2) \times (n + 2)$ containing the unknown interior temperature values and the known boundary values. Do not form the full matrix A_h (because it's sparse and this would waste memory); instead use the component form of Jacobi's method which has only five nonzero entries for each equation. Present the results as follows. Include a copy of the code.

a) For each value of h, plot the computed temperature w_{ij} at the final step using a contour plot and a mesh plot (type `help contour` and `help mesh` in Matlab for instructions). Use the `subplot` command to get several graphs on one plot, as in the lecture notes.

b) For each value of h, give the number of iterations required to reach the stopping criterion.

c) Give a brief writeup describing your results.