functions : $f(x) = \dfrac{1}{x+1} - \dfrac{1}{x-1}$, $x \to \infty$

integrals : $\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} dx$, $k \to \infty$

ODEs : $\dfrac{dy}{dt} = f(y)$, $t \to \infty$

PDE : $u_t = f(u) + \epsilon u_{xx}$, $\epsilon \to 0$

fluid dynamics : $u_t + u \cdot \nabla u = -\nabla p + \dfrac{1}{Re} \Delta u$, $Re \to 0, \infty$

1.1, 1.2 asymptotic expansions

def

1. $f(z) = O(g(z))$ as $z \to z_0$ in D $\Leftrightarrow$ $\left|\dfrac{f(z)}{g(z)}\right|$ is bounded as $z \to z_0$

2. $f(z) = o(g(z))$ as $z \to z_0$ in D $\Leftrightarrow$ $\left|\dfrac{f(z)}{g(z)}\right| \to 0$ as $z \to z_0$

ex

$\sin z = O(1)$ as $z \to 0$

$\sin z = o(1)$

$\sin z = O(z)$

$\sin z \neq o(z)$

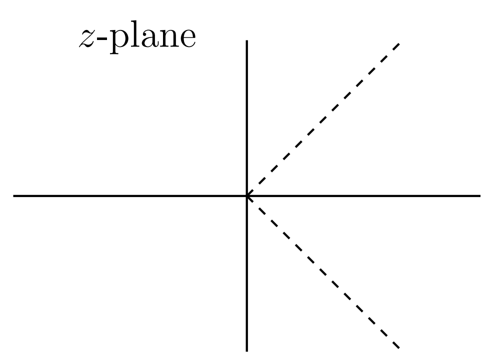
$\sin z = z + O(z^3)$

note : We can also consider $z \to \infty$.

ex

e^{-z} , $D = \{z : |z| > 0 \, , \, |\arg z| < \frac{\pi}{4}\}$

Then $e^{-z} = o(z^{-n})$ as $z \to \infty$ in D for all $n \geq 0$.



pf

$z = x + iy$, $z \in D \Rightarrow x > 0$

$\left|\dfrac{e^{-z}}{z^{-n}}\right| = e^{-x}(x^2+y^2)^{n/2} \leq e^{-x}(2x^2)^{n/2} = 2^{n/2} e^{-x} x^n \to 0$ as $z \to \infty$ in D ok

<u>def</u>

$f(z)$ is <u>asymptotic</u> to $g(z)$ as $z \to z_0$ in $D \Leftrightarrow \lim\limits_{z \to z_0} \dfrac{f(z)}{g(z)} = 1$

In this case we write $f(z) \sim g(z)$ as $z \to z_0$.

<u>ex</u>

$\sin z \sim z$ as $z \to 0$

$\sin z - z \sim -\frac{1}{3!} z^3$ as $z \to 0$

$\sinh z \sim \frac{1}{2} e^z$ as $z \to \infty$, $|\arg z| < \frac{\pi}{4}$

$\sinh z \sim \frac{1}{2} e^{-z}$ as $z \to \infty$, $|\arg z - \pi| < \frac{\pi}{4}$

<u>def</u>

$f(z)$ is <u>analytic</u> at $z = z_0 \Leftrightarrow f(z) = \sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$ for $|z - z_0| < R$

$\Leftrightarrow f(z) = \lim\limits_{N \to \infty} s_N(z)$ for $|z - z_0| < R$, where $s_N(z) = \sum\limits_{n=0}^{N} a_n (z - z_0)^n$

$\Leftrightarrow f(z) =$ convergent power series

note : We can also consider $z_0 = \infty$.

<u>ex</u>

$$\frac{1}{z^2+1} = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - \cdots \text{ in } |z| < 1 \quad , \quad z_0 = 0$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+2}} = \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \cdots \text{ in } |z| > 1 \quad , \quad z_0 = \infty$$

$$\frac{1}{z^2+1} \sim 1 \text{ as } z \to 0$$

$$\frac{1}{z^2+1} - 1 \sim -z^2 \text{ as } z \to 0$$

$$\frac{1}{z^2+1} \sim \frac{1}{z^2} \text{ as } z \to \infty$$

$$\frac{1}{z^2+1} - \frac{1}{z^2} \sim -\frac{1}{z^4} \text{ as } z \to \infty$$

1. Here there is no restriction on $z \to \infty$.

2. Convergent series are not the only examples of asymptotic relations.

<u>ex</u>

$\mathrm{Ei}(x) = \int_x^\infty e^{-t} t^{-1} dt$: <u>exponential integral</u> , improper , converges for $x > 0$

<u>pf</u>

$$\int_x^\infty e^{-t} t^{-1} dt \le \int_x^\infty e^{-t} x^{-1} dt = -e^{-t} x^{-1} \Big|_x^\infty = \frac{e^{-x}}{x} < \infty \quad \underline{\text{ok}}$$

This shows that $\lim_{x\to\infty} \mathrm{Ei}(x) = 0$, but precisely how fast does $\mathrm{Ei}(x) \to 0$ as $x \to \infty$?

$$\begin{aligned}\mathrm{Ei}(x) = \int_x^\infty e^{-t} t^{-1} dt &= -e^{-t} t^{-1} \Big|_x^\infty - \int_x^\infty e^{-t} t^{-2} dt \ : \ \text{integration by parts} \\ &= \frac{e^{-x}}{x} - \int_x^\infty e^{-t} t^{-2} dt \\ &= \frac{e^{-x}}{x} - \left(-e^{-t} t^{-2} \Big|_x^\infty - 2\int_x^\infty e^{-t} t^{-3} dt \right) \\ &= \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + 2\int_x^\infty e^{-t} t^{-3} dt \quad \ldots\end{aligned}$$

$\mathrm{Ei}(x) = s_n(x) + r_n(x)$

$$s_n(x) = e^{-x} \left(\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{3!}{x^4} + \cdots + \frac{(-1)^{n+1}(n-1)!}{x^n} \right)$$

$$r_n(x) = (-1)^n n! \int_x^\infty e^{-t} t^{-(n+1)} dt$$

1. $\lim_{n\to\infty} s_n(x) = \sum_{n=1}^\infty \frac{(-1)^{n+1}(n-1)!}{x^n}$ diverges for all x , pf ...

2. Fix $n \ge 1$. Then $|r_n(x)| \le n! \frac{e^{-x}}{x^{n+1}} \to 0$ as $x \to \infty$.

Hence $s_n(x)$ is an approximation to $\mathrm{Ei}(x)$ for large x.

In fact, it can be shown that $\frac{\mathrm{Ei}(x)}{s_n(x)} = 1 + \frac{r_n(x)}{s_n(x)} \to 1$ as $x \to \infty$ (hw),

so $\mathrm{Ei}(x) \sim s_n(x)$ as $x \to \infty$ for all $n \ge 1$; this answers the question above.

3. Even though the series diverges for all x, the partial sums $s_n(x)$ approximate $\mathrm{Ei}(x)$ better and better as $x \to \infty$.

question

Given x, we want to approximate $\mathrm{Ei}(x)$ by $s_n(x)$; what is the best choice of n?

ex : $\mathrm{Ei}(5) = 0.001148$

n	$s_n(5)$	
1	0.001348	
2	0.001078	
3	0.001186	
4	0.001121	
5	0.001173	$\leftarrow$
6	0.001121	
7	0.001183	
14	-0.003846	
19	1.775902	

answer

The best choice of n is the one that minimizes $|r_n(x)|$.

note : we know that $|r_n(x)| \leq n!\dfrac{e^{-x}}{x^{n+1}}$, but in fact $|r_n(x)| \sim n!\dfrac{e^{-x}}{x^{n+1}}$ as $x \to \infty$

pf : hw

$\Rightarrow$ the error in $s_n(x) \sim$ the 1st neglected term in the series as $x \to \infty$

$$\Rightarrow \frac{r_n(x)}{r_{n-1}(x)} \sim \frac{n!\dfrac{e^{-x}}{x^{n+1}}}{(n-1)!\dfrac{e^{-x}}{x^n}} = \frac{n}{x} \leq 1 \Leftrightarrow n \leq x$$

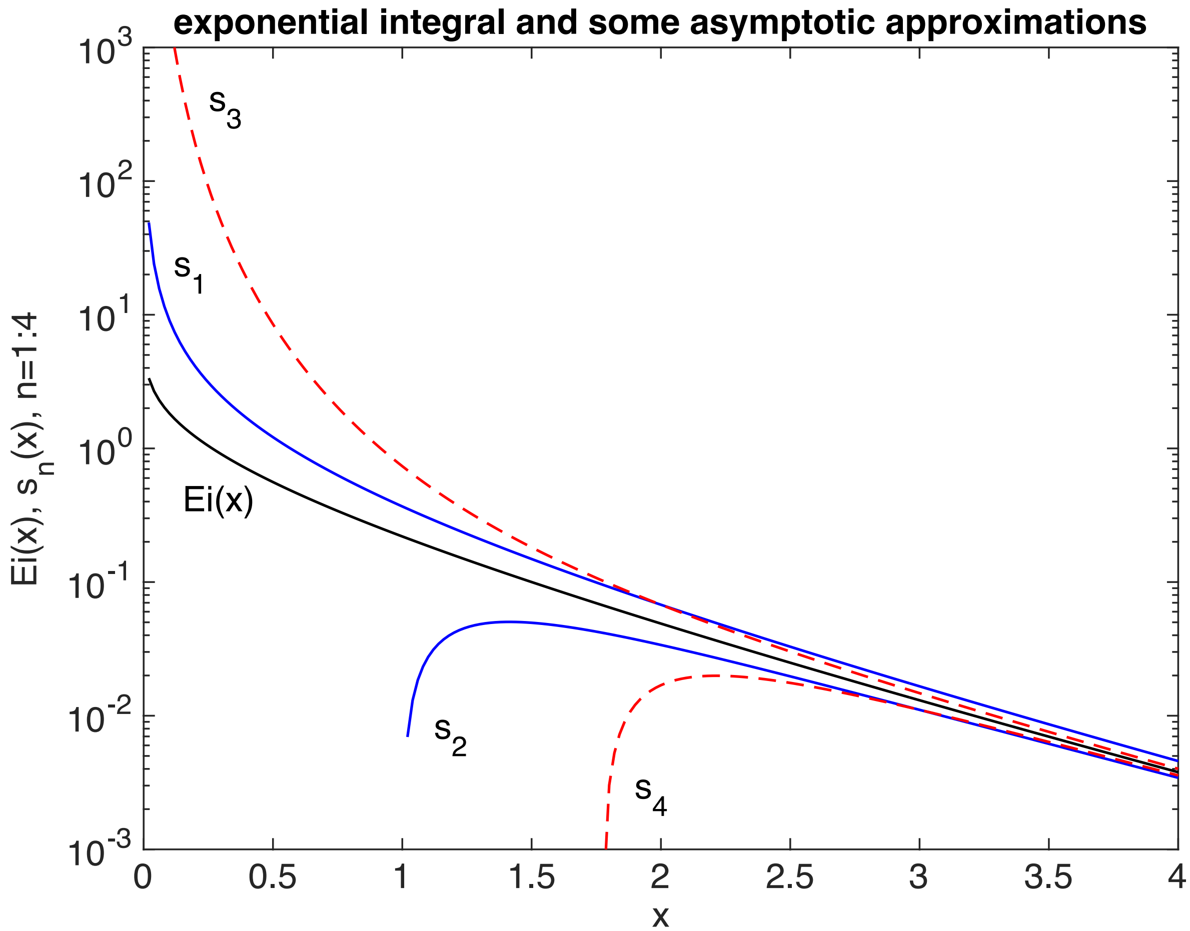
$\Rightarrow$ the best choice is $n = n(x) = [x]$: largest integer $\leq x$

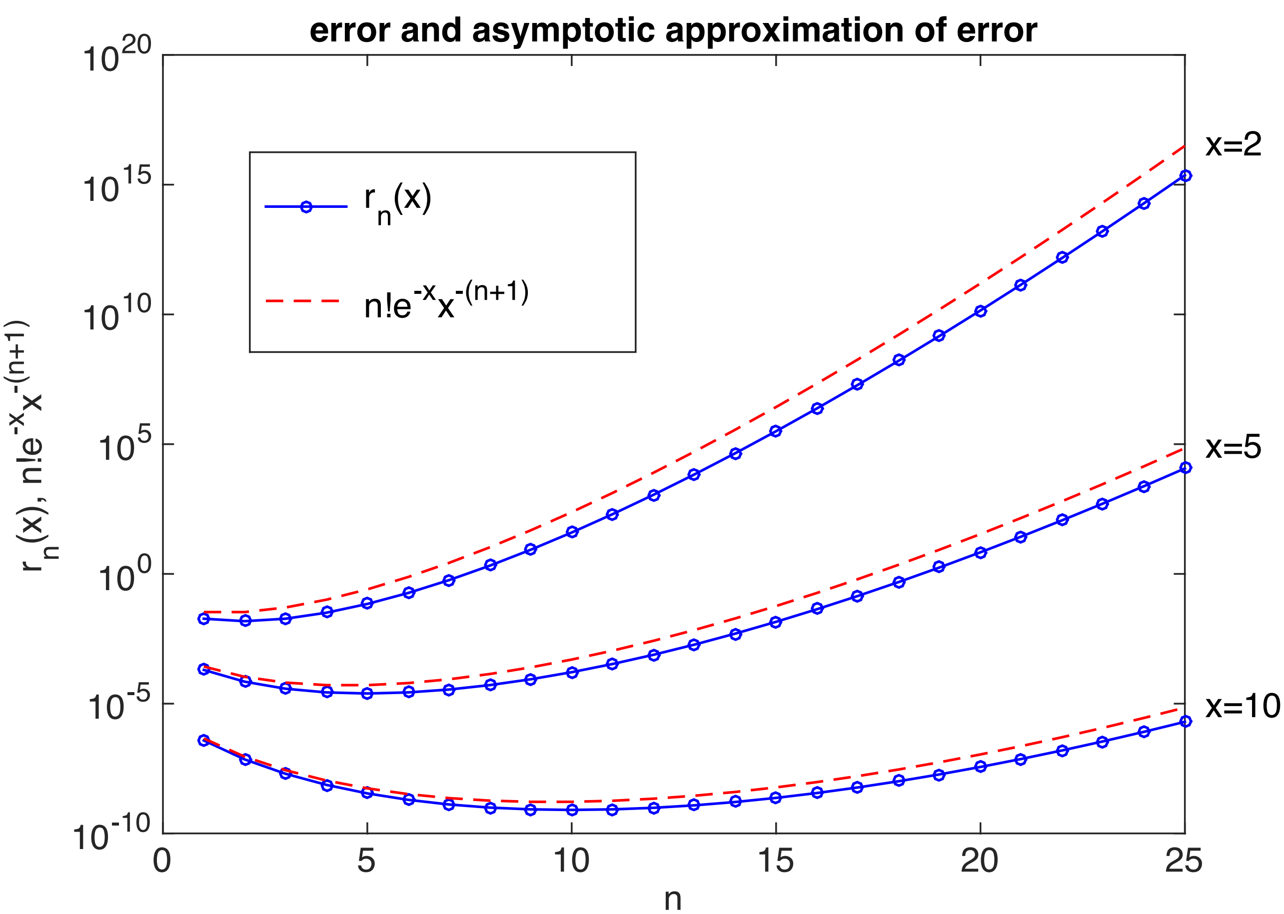
Given x, the error cannot be made arbitrarily small by increasing n; however as x increases,

1. the minimum error decreases,

2. the optimal n increases,

3. the error versus n curve becomes flatter near the minimum, so the error is small even for $n << [x]$, e.g. even $n = 1$ may be adequate in some applications.


exponential integral and some asymptotic approximations
Ei(x), s_n(x), n=1:4
x
s_3
s_1
Ei(x)
s_2
s_4


error and asymptotic approximation of error
r_n(x), n!e^{-x}x^{-(n+1)}
n
r_n(x)
n!e^{-x}x^{-(n+1)}
x=2
x=5
x=10

def: $\{\phi_n(z) : n = 1, 2, \ldots\}$ is an asymptotic sequence as $z \to z_0$

$$\Leftrightarrow \ \phi_{n+1}(z) = o(\phi_n(z)) \text{ as } z \to z_0 \ \Leftrightarrow \ \lim_{z \to z_0} \frac{\phi_{n+1}(z)}{\phi_n(z)} = 0$$

ex : $\phi_n(z) = (z - z_0)^n$ as $z \to z_0$

$\phi_n(z) = e^{-nz}$ as $z \to \infty$

$\phi_n(z) = z^n \log z$ as $z \to 0$

def: $f(z) \sim \sum_{n=1}^{\infty} a_n \phi_n(z)$ as $z \to z_0$: asymptotic expansion wrt $\{\phi_n(z)\}$

$$\Leftrightarrow \ f(z) = \sum_{n=1}^{N} a_n \phi_n(z) + o(\phi_N(z)) \text{ as } z \to z_0 \text{ for all } N \geq 1$$

$$\Leftrightarrow \ f(z) = \sum_{n=1}^{N} a_n \phi_n(z) + O(\phi_{N+1}(z)) \text{ as } z \to z_0 \text{ for all } N \geq 1$$

In this case, the error has the same order of magnitude as the first term omitted.

ex 1 : If $f(z)$ is analytic at $z = z_0$, then $f(z) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ as $z \to z_0$.

Hence a convergent power series is also an asymptotic expansion.

ex 2 : $\text{Ei}(x) = \int_x^{\infty} e^{-t} t^{-1} dt \ \sim \ e^{-x} \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}(n-1)!}{x^n}$ as $x \to \infty$

pf : $\phi_n(x) = e^{-x} x^{-n}$: asymptotic sequence as $x \to \infty$

$$\left| \frac{\text{Ei}(x) - s_n(x)}{\phi_n(x)} \right| = \left| \frac{r_n(x)}{\phi_n(x)} \right| \leq \frac{n! e^{-x} x^{-(n+1)}}{e^{-x} x^{-n}} = \frac{n!}{x} \to 0 \text{ as } x \to \infty \qquad \text{ok}$$

Hence even though the series diverges, it is still an asymptotic expansion.

properties of asymptotic expansions

1. If $f(z) \sim \sum_{n=1}^{\infty} a_n \phi_n(z)$ as $z \to z_0$, then the a_n are unique.

pf : $f(z) = a_1 \phi_1(z) + o(\phi_1(z)) \Rightarrow \lim_{z \to z_0} \frac{f(z)}{\phi_1(z)} = a_1$

$$f(z) = a_1 \phi_1(z) + a_2 \phi_2(x) + o(\phi_2(z)) \Rightarrow \lim_{z \to z_0} \frac{f(z) - a_1 \phi_1(z)}{\phi_2(z)} = a_2 \ , \ \ldots \qquad \text{ok}$$

2. If $f(z) \sim \sum_{n=1}^{\infty} \frac{a_n}{z^n}$ as $z \to \infty$, then $f(z) + e^{-z} \sim \sum_{n=1}^{\infty} \frac{a_n}{z^n}$ as $z \to \infty$.

Hence two different functions can have the same asymptotic expansion.

3. $\frac{1}{z-1} \sim \sum_{n=1}^{\infty} \frac{1}{z^n}$ as $z \to \infty$, $\frac{1}{z-1} \sim (z+1) \sum_{n=1}^{\infty} \frac{1}{z^{2n}}$ as $z \to \infty$, pf ...

Hence a function can have two different asymptotic expansions.