2.1 Watson's lemma

recall : $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$: Gamma function

properties : $\Gamma(n+1) = n!$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

pf : $\Gamma(n+1) = \int_0^\infty e^{-t} t^n dt = -e^{-t} t^n \Big|_0^\infty + \int_0^\infty e^{-t} n t^{n-1} dt \Rightarrow \Gamma(n+1) = n\Gamma(n) \ \ldots$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-t} t^{-1/2} dt \ , \ \text{set } t = s^2 \, , \ dt = 2sds$$

$$= \int_0^\infty e^{-s^2} s^{-1} 2sds = 2\int_0^\infty e^{-s^2} ds = \int_{-\infty}^\infty e^{-s^2} ds \ \ldots \quad \underline{\text{ok}}$$

ex : $\int_0^\infty e^{-xt} t^\lambda dt$: Laplace transform , assume $\lambda > -1$, consider $x \to \infty$

integration by parts fails

$$\int_0^\infty e^{-xt} t^\lambda dt = \frac{e^{-xt}}{-x} t^\lambda \Big|_0^\infty - \int_0^\infty \frac{e^{-xt}}{-x} \lambda t^{\lambda-1} dt \ : \ \text{diverges , try other way}$$

$$\int_0^\infty e^{-xt} t^\lambda dt = e^{-xt} \frac{t^{\lambda+1}}{\lambda+1} \Big|_0^\infty - \int_0^\infty -xe^{-xt} \frac{t^{\lambda+1}}{\lambda+1} dt \ : \ \text{not asymptotic as } x \to \infty$$

change of variables : set $s = xt$, $ds = xdt$

$$\int_0^\infty e^{-xt} t^\lambda dt = \int_0^\infty e^{-s} \left(\frac{s}{x}\right)^\lambda \frac{ds}{x} = \frac{1}{x^{\lambda+1}} \int_0^\infty e^{-s} s^\lambda ds = \frac{\Gamma(\lambda+1)}{x^{\lambda+1}} \ : \ \text{exact for all } x > 0$$

Watson's lemma : Assume $\lambda > -1$, $g(t)$ is analytic at $t = 0$, $g(0) \neq 0$.

$$\int_0^T e^{-xt} t^\lambda g(t) dt \sim \sum_{n=0}^\infty \frac{g^{(n)}(0)}{n!} \frac{\Gamma(\lambda+n+1)}{x^{\lambda+n+1}} \ \text{as } x \to \infty$$

1. the expansion only depends on $g(t)$ near $t = 0$ and is independent of T
2. previous example is $T = \infty$, $g(t) = 1$
3. larger λ implies faster decay as $x \to \infty$

case 1 : $\int_0^T e^{-xt} t^\lambda dt \sim \frac{\Gamma(\lambda+1)}{x^{\lambda+1}}$ as $x \to \infty$

$g(t) = 1$, set $s = xt$, $ds = xdt$

$$\int_0^T e^{-xt} t^\lambda dt = \int_0^\infty e^{-xt} t^\lambda dt - \int_T^\infty e^{-xt} t^\lambda dt = \frac{\Gamma(\lambda+1)}{x^{\lambda+1}} - \int_{xT}^\infty e^{-s} \left(\frac{s}{x}\right)^\lambda \frac{ds}{x}$$

$$= \frac{\Gamma(\lambda+1)}{x^{\lambda+1}} - \frac{1}{x^{\lambda+1}} \int_{xT}^\infty e^{-s} s^\lambda ds \ : \ \text{will show the error is } o(e^{-xT}) \text{ as } x \to \infty$$

set $s = xT(1+u)$, $ds = xTdu$, assume $xT > \lambda$

$$\int_{xT}^\infty e^{-s} s^\lambda ds = \int_0^\infty e^{-xT(1+u)} (xT(1+u))^\lambda xTdu = e^{-xT} (xT)^{\lambda+1} \int_0^\infty e^{-xTu} (1+u)^\lambda du$$

$$\lambda \le 0 \Rightarrow \int_0^\infty e^{-xTu}(1+u)^\lambda du \le \int_0^\infty e^{-xTu} du = \frac{e^{-xTu}}{-xT}\bigg|_0^\infty = \frac{1}{xT}$$

$$\lambda > 0 \Rightarrow \int_0^\infty e^{-xTu}(1+u)^\lambda du \le \int_0^\infty e^{-xTu} e^{\lambda u} du = \frac{e^{(-xT+\lambda)u}}{-xT+\lambda}\bigg|_0^\infty = \frac{1}{xT-\lambda} \quad \underline{\text{ok}}$$

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case 2 : $g(t) = \sum_{n=0}^{N} \frac{g^{(n)}(0)}{n!} t^n + r_N(t)$, $|r_N(t)| \le L t^{N+1}$ for $|t| < R$

assume $T < R$ (won't affect final result)

$$\int_0^T e^{-xt} t^\lambda g(t) dt = \int_0^T e^{-xt} t^\lambda \sum_{n=0}^{N} \frac{g^{(n)}(0)}{n!} t^n dt + \int_0^T e^{-xt} t^\lambda r_N(t) dt = a + b$$

$$a = \sum_{n=0}^{N} \frac{g^{(n)}(0)}{n!} \int_0^T e^{-xt} t^{\lambda+n} dt = \sum_{n=0}^{N} \frac{g^{(n)}(0)}{n!} \frac{\Gamma(\lambda+n+1)}{x^{\lambda+n+1}} + o(e^{-xT}) \text{ as } x \to \infty$$

$$|b| \le L \int_0^T e^{-xt} t^{\lambda+N+1} dt = O(x^{-(\lambda+N+2)}) \text{ as } x \to \infty \quad \underline{\text{ok}}$$

note : if $T > R$, need to also assume $|g(t)| \le K e^{ct}$ for $0 \le t \le T$ and consider $\int_0^T = \int_0^{T_1} + \int_{T_1}^T$, where $T_1 < R$

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ex 1 : $\int_0^\infty e^{-xt} \log(1+t^2) dt \sim \frac{2}{x^3} - \frac{12}{x^5} + \cdots$ as $x \to \infty$

$\log(1+t^2) = t^2 - \frac{1}{2}t^4 + \cdots = t^2(1 - \frac{1}{2}t^2 + \cdots)$ for $|t| < 1$

$\lambda = 2$, $g(t) = 1 - \frac{1}{2}t^2 + \cdots$, $\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \frac{\Gamma(\lambda+n+1)}{x^{\lambda+n+1}} = \frac{\Gamma(3)}{x^3} - \frac{1}{2}\frac{\Gamma(5)}{x^5} + \cdots$

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ex 2 : $\int_0^\infty \frac{e^{-t}}{t+x} dt \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{x^n} = \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \cdots$ as $x \to \infty$

method 1 : $s = t + x \Rightarrow \int_x^\infty \frac{e^{-(s-x)}}{s} ds = e^x \int_x^\infty e^{-s} s^{-1} ds = e^x \cdot \mathrm{Ei}(x)$

method 2 : $t = xs \Rightarrow \int_0^\infty \frac{e^{-xs}}{xs+x} x ds = \int_0^\infty e^{-xs}(1+s)^{-1} ds$: Watson's lemma

$\lambda = 0$, $g(s) = (1+s)^{-1} = \sum_{n=0}^{\infty} (-1)^n s^n$ for $|s| < 1$

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \frac{\Gamma(\lambda+n+1)}{x^{\lambda+n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+1)}{x^{n+1}} \quad \underline{\text{ok}}$$

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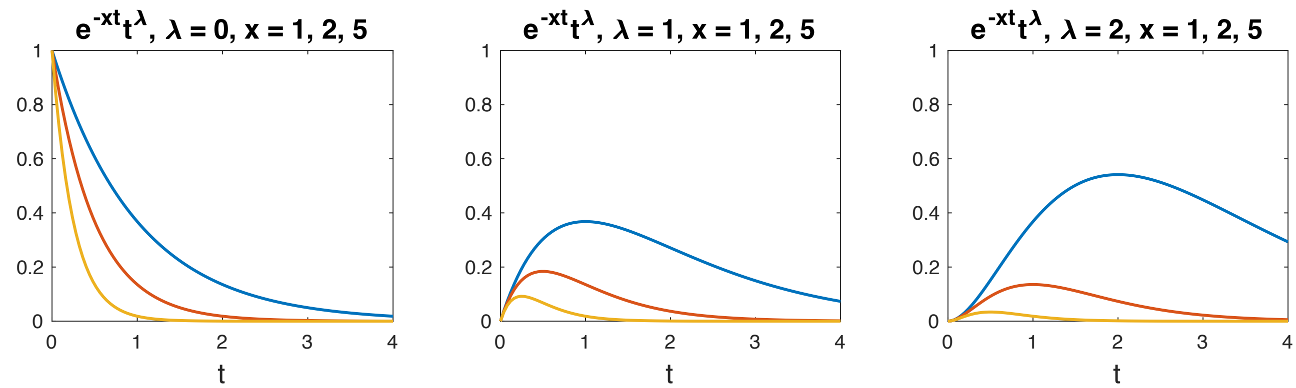
ex 3 : $\int_0^\infty e^{-xt^2} t^\lambda dt = \frac{\Gamma((\lambda+1)/2)}{2x^{(\lambda+1)/2}}$ for all $x > 0$

pf : $s = t^2$, $ds = 2t dt = 2s^{1/2} dt$, $\int_0^\infty e^{-xs} s^{\lambda/2} \frac{ds}{2s^{1/2}} = \frac{1}{2} \int_0^\infty e^{-xs} s^{(\lambda-1)/2} ds \quad \underline{\text{ok}}$

(a) $\int_0^\infty e^{-xt} t^\lambda dt = \frac{\Gamma(\lambda+1)}{x^{\lambda+1}}$ for all $x > 0$

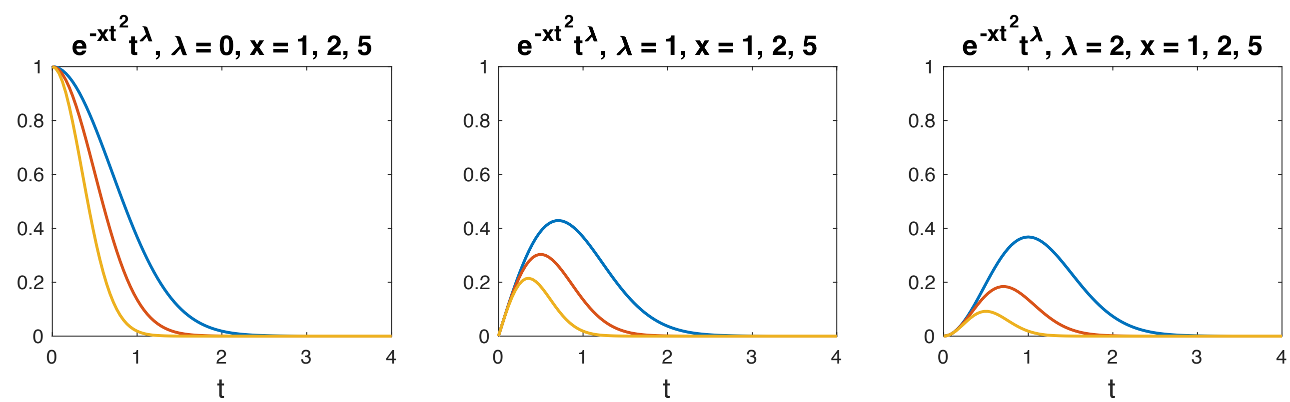


note : larger $\lambda \Rightarrow$ the integral decays faster as $x \to \infty$

Watson's lemma : $\int_0^T e^{-xt} t^\lambda g(t) dt \sim \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} \frac{\Gamma(\lambda+n+1)}{x^{\lambda+n+1}}$ as $x \to \infty$

"…all contributions to the asymptotic approximation as $x \to \infty$ come from the region near $t = 0$ irrespective of the order of the zero of $t^\lambda g(t)$ …"

(b) $\int_0^\infty e^{-xt^2} t^\lambda dt = \frac{\Gamma((\lambda+1)/2)}{2x^{(\lambda+1)/2}}$ for all $x > 0$



note : integral (b) decays slower than integral (a) as $x \to \infty$

extension of Watson's lemma : $\int_{-\alpha}^{\beta} e^{-xt^2} t^\lambda g(t) dt \sim \ldots$ as $x \to \infty$

"Again all of the contributions to the asymptotic aproximation come from the neighborhood of $t = 0$."

2.2 Laplace's method

$\int_\alpha^\beta e^{xh(t)}g(t)dt\ ,\ x\to\infty$

Watson's lemma has $\alpha=0, h(t)=-t$.

idea : The main contribution comes from t_0 such that $h(t_0)=\max\limits_{\alpha\le t\le\beta} h(t)$.

ex: $\int_0^\infty e^{-x\cosh t}dt \sim e^{-x}\left(\frac{\pi}{2x}\right)^{1/2}$ as $x\to\infty$

$h(t)=-\cosh t \Rightarrow t_0=0$

$\cosh t\sim 1$ as $t\to 0 \Rightarrow f(x)\sim\int_0^\infty e^{-x}dt=e^{-x}\cdot\infty$: fails

$\cosh t\sim 1+\frac{1}{2}t^2$ as $t\to 0 \Rightarrow f(x)\sim\int_0^\infty e^{-x(1+\frac{1}{2}t^2)}dt = e^{-x}\int_0^\infty e^{-\frac{1}{2}xt^2}dt$

$s^2=\frac{1}{2}xt^2\,,\,2sds=xtdt \Rightarrow e^{-x}\int_0^\infty e^{-s^2}\frac{2sds}{x(2s^2/x)^{1/2}} = e^{-x}\left(\frac{2}{x}\right)^{1/2}\int_0^\infty e^{-s^2}ds = \cdots$

how to find the next term?

$\cosh t\sim 1+\frac{1}{2}t^2+\frac{1}{24}t^4$ as $t\to 0 \Rightarrow \int_0^\infty e^{-x\cosh t}dt\sim\int_0^\infty e^{-x(1+\frac{1}{2}t^2+\frac{1}{24}t^4)}dt$

$s^2=x(\frac{1}{2}t^2+\frac{1}{24}t^4)\,,\,2sds=x(t+\frac{1}{6}t^3)dt$: cumbersome

alternative : Laplace's method

$\cosh t=1+s^2\,,\,\sinh t dt=2sds\,,\,\sinh t=(\cosh^2 t-1)^{1/2}=(2s^2+s^4)^{1/2}$

$$\int_0^\infty e^{-x\cosh t}dt = \int_0^\infty e^{-x(1+s^2)}\frac{2sds}{(2s^2+s^4)^{1/2}} = 2e^{-x}\int_0^\infty e^{-xs^2}(2+s^2)^{-1/2}ds$$

could set $s^2=r$ and apply Watson's lemma ...

instead consider $(2+s^2)^{-1/2}=(2(1+\frac{1}{2}s^2))^{-1/2}=2^{-1/2}(1-\frac{1}{4}s^2+O(s^4))$

$$\int_0^\infty e^{-x\cosh t}dt\sim 2\cdot 2^{-1/2}\,e^{-x}\int_0^\infty e^{-xs^2}(1-\tfrac{1}{4}s^2)ds$$

$$\int_0^\infty e^{-xs^2}ds=\frac{1}{2}\left(\frac{\pi}{x}\right)^{1/2}\ ,\ \int_0^\infty e^{-xs^2}s^2ds=\frac{\sqrt{\pi}}{4}x^{-3/2}=\frac{1}{2}\left(\frac{\pi}{x}\right)^{1/2}\cdot\frac{1}{2x}$$

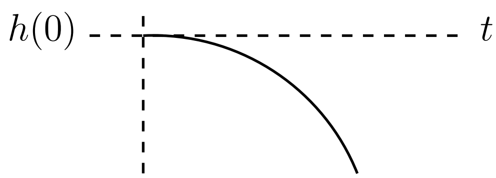
$\int_0^\infty e^{-x\cosh t}dt\sim\left(\frac{\pi}{2x}\right)^{1/2}e^{-x}\left(1-\frac{1}{8x}\right)$ as $x\to\infty$, next term : hw2

note : the same result holds for $\int_0^T e^{-x\cosh t}dt$ for all $T>0$, why? : hw2

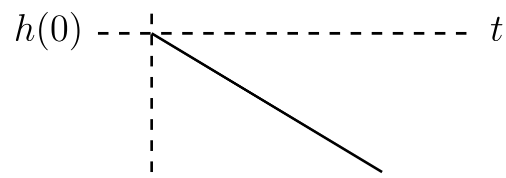
<u>general form</u>

wlog consider $\int_0^T e^{xh(t)} g(t)dt$, where $h(0) = \max\limits_{0 \le t \le T} h(t)$

<u>case 1</u> : $h'(0) = 0$, $h''(0) < 0$



<u>case 2</u> : $h'(0) < 0$



case 1 : $h(t) = h(0) - s^2$: solve for $t = t(s)$, implicit function theorem

$$h(t) = \cancel{h(0)} + \cancel{h'(0)}t + \tfrac{1}{2}h''(0)t^2 + O(t^3) = \cancel{h(0)} - s^2$$

$$\Rightarrow \tfrac{1}{2}h''(0)t^2 + O(t^3) = -s^2 \Rightarrow t = \left(\frac{-2}{h''(0)}\right)^{1/2} s + O(s^2)$$

$$\int_0^T e^{xh(t)} g(t)dt \sim \int_0^\epsilon e^{xh(t)} g(t)dt \sim \int_0^\delta e^{x(h(0)-s^2)}(g(0)+O(s))\left(\left(\frac{-2}{h''(0)}\right)^{1/2} + O(s)\right)ds$$

$$\sim g(0)\left(\frac{-2}{h''(0)}\right)^{1/2} e^{xh(0)}\int_0^\infty e^{-xs^2}ds + e^{xh(0)}O\Big(\int_0^\infty e^{-xs^2}s\,ds\Big)$$

$$\int_0^\infty e^{-xs^2}ds = \frac{1}{2}\left(\frac{\pi}{x}\right)^{1/2} \ , \ \int_0^\infty e^{-xs^2}s\,ds = \frac{1}{2x} \ : \ \text{check} \ldots$$

$$\int_0^T e^{xh(t)} g(t)dt \sim g(0)\left(\frac{-\pi}{2h''(0)x}\right)^{1/2} e^{xh(0)} + e^{xh(0)}O(x^{-1}) \text{ as } x \to \infty$$

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case 2 : $h(t) = h(0) - s$

$$h(t) = \cancel{h(0)} + h'(0)t + O(t^2) = \cancel{h(0)} - s \text{ as } s,t \to 0$$

$$\Rightarrow h'(0)t + O(t^2) = -s \Rightarrow t = \frac{-1}{h'(0)}s + O(s^2)$$

$$\int_0^T e^{xh(t)} g(t)dt \sim \cdots \sim \int_0^\infty e^{x(h(0)-s)}(g(0)+O(s))\left(\frac{-1}{h'(0)} + O(s)\right)ds$$

$$\sim \frac{-g(0)}{h'(0)}\, e^{xh(0)}\int_0^\infty e^{-xs}ds + e^{xh(0)}O\Big(\int_0^\infty e^{-xs}s\,ds\Big)$$

$$\int_0^\infty e^{-xs}ds = \frac{1}{x} \ , \ \int_0^\infty e^{-xs}s\,ds = \frac{1}{x^2} \ : \ \text{check} \ldots$$

$$\int_0^T e^{xh(t)} g(t)dt \sim \frac{-g(0)}{h'(0)x}\, e^{xh(0)} + e^{xh(0)}O(x^{-2}) \text{ as } x \to \infty$$

note : case 2 decays faster than case 1 as $x \to \infty$

recall : $\int_0^\infty e^{-x\cosh t}dt \sim e^{-x}\left(\frac{\pi}{2x}\right)^{1/2}$ as $x \to \infty$

$g(t) = 1\,,\ h(t) = -\cosh t \ \Rightarrow\ h(0) = -1\ ,\ h'(0) = 0\ ,\ h''(0) = -1\ :\ $ case 1

$\int_0^T e^{xh(t)}g(t)dt \sim g(0)\left(\frac{-\pi}{2h''(0)x}\right)^{1/2} e^{xh(0)}\ ,\ g(0) = 1$ ok

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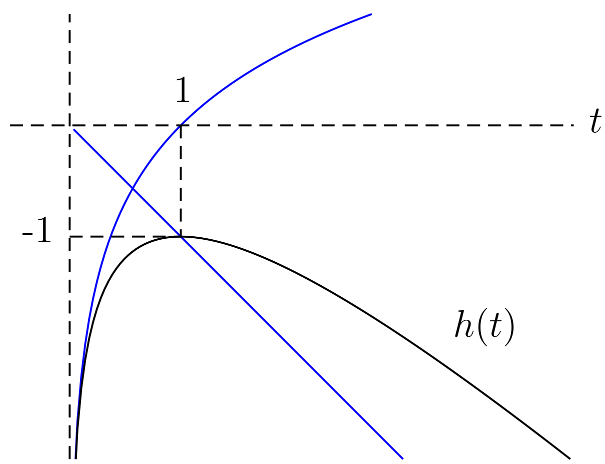
ex 1

$\Gamma(x+1) = \int_0^\infty e^{-t}t^x dt = \int_0^\infty e^{-t}e^{x\ln t}dt\ :\ $ cannot apply Laplace's method

set $t = xs\,,\ dt = xds$

$\Gamma(x+1) = \int_0^\infty e^{-xs+x(\ln x+\ln s)}x\,ds = xe^{x\ln x}\int_0^\infty e^{x(\ln s - s)}ds = x\cdot x^x\int_0^\infty e^{x(\ln t - t)}dt$

$h(t) = \ln t - t\ :\ $ case 1



$h'(t) = \frac{1}{t} - 1 = 0 \ \Rightarrow\ t_0 = 1$

$h''(t) = -\frac{1}{t^2} \ \Rightarrow\ h''(1) = -1$

$h(1) = -1 = \max_{t>0} h(t)$

$h(t) \sim \ln t$ as $t \to 0^+$

$h(t) = h(1) - s^2$

$\ln t - t = -1 - s^2\ $, solve for $t = t(s)$

$\cancel{h(1)} + \cancel{h'(1)}(t-1) + \frac{1}{2}h''(1)(t-1)^2 + O((t-1)^3) = \cancel{h(1)} - s^2$

$-\frac{1}{2}(t-1)^2 + O((t-1)^3) = -s^2\,,\ t = 1 \Leftrightarrow s = 0$

$t = a_0 + a_1 s + a_2 s^2 + \cdots = 1 + a_1 s + a_2 s^2 + \cdots \Rightarrow t - 1 = s(a_1 + a_2 s + \cdots)$

$-\frac{1}{2}s^2(a_1 + a_2 s + \cdots)^2 + O(s^3) = -s^2$

$-\frac{1}{2}a_1^2 = -1 \Rightarrow a_1^2 = 2 \Rightarrow a_1 = \pm\sqrt{2}$, choose $+ \Rightarrow t = 1 + \sqrt{2}s + O(s^2)$

$$\Gamma(x+1) \sim x^{x+1}\int_{1-\epsilon_1}^{1+\epsilon_2} e^{x(\ln t - t)}dt \sim x^{x+1}\int_{-\delta_1}^{\delta_2} e^{-x(1+s^2)}\sqrt{2}\,ds$$

$$\sim x^{x+1}e^{-x}\sqrt{2}\int_{-\infty}^\infty e^{-xs^2}ds = x^{x+1}e^{-x}\sqrt{2}\left(\frac{\pi}{x}\right)^{1/2}$$

$\Gamma(x+1) \sim \sqrt{2\pi}\,x^{x+\frac{1}{2}}e^{-x}$ as $x \to \infty$, find the next term : hw2

$\Gamma(n+1) = n! \sim \sqrt{2\pi}\,n^{n+\frac{1}{2}}e^{-n}$ as $n \to \infty$: Stirling's formula

<u>ex 2</u>

$f(x) = \int_0^{\pi/2} e^{x\cos t} \ln(\lambda + \sin t)dt \;,\; \lambda > 0$

$h(t) = \cos t \;,\; h(0) = 1 = \max\limits_{0\le t\le\pi/2} h(t) \;,\; h'(0) = 0 \;,\; h''(0) = -1 \;:\;$ case 1

$h(t) = h(0) - s^2 \Rightarrow \cos t = 1 - s^2 \;,$ solve for $t = t(s)$

$\cancel{h(0)} + \cancel{h'(0)}t + \frac{1}{2}h''(0)t^2 + \frac{1}{3!}\cancel{h'''(0)}t^3 + O(t^4) = \cancel{h(0)} - s^2$

$-\frac{1}{2}t^2 + O(t^4) = -s^2 \,,\, t = 0 \Leftrightarrow s = 0$

$t = \cancel{a_0} + a_1 s + a_2 s^2 + \cdots = s(a_1 + a_2 s + \cdots) = \ldots = \sqrt{2}s + O(s^3) \;,$ check ...

$g(t) = \ln(\lambda + \sin t) \;\Rightarrow\; g(0) = \ln\lambda$

<u>$\lambda \neq 1$</u>

$f(x) \;\sim \int_0^{\infty} e^{x(1-s^2)} \ln\lambda \cdot \sqrt{2}\, ds \;=\; e^x \ln\lambda \cdot \sqrt{2} \int_0^{\infty} e^{-xs^2} ds$

$\int_0^{\pi/2} e^{x\cos t} \ln(\lambda + \sin t)dt \;\sim\; \left(\frac{\pi}{2x}\right)^{1/2} e^x \ln\lambda$ as $x \to \infty$: fails for $\lambda = 1$

<u>$\lambda = 1$</u>

$g(t) = \ln(1 + \sin t) \;\Rightarrow\; g(0) = 0 \;,\; g'(t) = \dfrac{\cos t}{1 + \sin t} \;\Rightarrow\; g'(0) = 1$

$g(t) = \cancel{g(0)} + g'(0)t + O(t^2) = t + O(t^2) = \sqrt{2}\, s + O(s^2)$

$f(x) \;\sim \int_0^{\infty} e^{x(1-s^2)} \sqrt{2}\, s \cdot \sqrt{2}\, ds \;= 2e^x \int_0^{\infty} e^{-xs^2} s\, ds$

$\int_0^{\pi/2} e^{x\cos t} \ln(1 + \sin t)dt \;\sim\; \dfrac{e^x}{x}$ as $x \to \infty$

$\lim\limits_{\lambda\to 1} \lim\limits_{x\to\infty} \neq \lim\limits_{x\to\infty} \lim\limits_{\lambda\to 1}$: to clarify this, consider the next term in the case $\lambda \neq 1$

$g(t) = \ln(\lambda + \sin t) \;\Rightarrow\; g(0) = \ln\lambda \;,\; g'(t) = \dfrac{\cos t}{\lambda + \sin t} \;\Rightarrow\; g'(0) = \dfrac{1}{\lambda}$

$g(t) = g(0) + g'(0)t + O(t^2) = \ln\lambda + \dfrac{t}{\lambda} + O(t^2) = \ln\lambda + \dfrac{\sqrt{2}\, s}{\lambda} + O(s^2)$

$$\int_0^{\pi/2} e^{x\cos t} \ln(\lambda + \sin t)dt \sim \int_0^{\infty} e^{x(1-s^2)} \Big(\ln\lambda + \frac{\sqrt{2}\, s}{\lambda} + O(s^2)\Big)\Big(\sqrt{2} + O(s^2)\Big) ds$$

$$\sim\; e^x \int_0^{\infty} e^{-xs^2} \Big(\sqrt{2}\ln\lambda + \frac{2s}{\lambda} + O(s^2)\Big) ds$$

$$\sim\; e^x \left(\left(\frac{\pi}{2x}\right)^{1/2} \ln\lambda + \frac{1}{\lambda x} + O\Big(x^{-3/2}\Big)\right) \text{ as } x \to \infty \;:\; \text{now we can let } \lambda \to 1$$