3.1 method of steepest descent

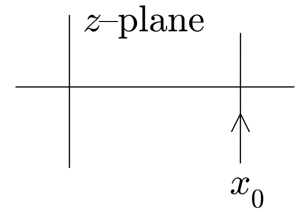
$f(\lambda) = \int_C g(z) e^{\lambda h(z)} dz$, $\lambda \to \infty$, assume $g(z)$, $h(z)$ are analytic

$h(z) = \phi(z) + i\psi(z)$, $e^{\lambda h(z)} = e^{\lambda\phi(z)} \cdot e^{i\lambda\psi(z)} = e^{\lambda\phi(z)}(\cos(\lambda\psi(z)) + i\sin(\lambda\psi(z)))$

2 effects : $\phi \to$ exponential growth/decay , $\psi \to$ oscillation/cancellation

ex 0

$f(\lambda) = \int_C e^{\lambda z^2} dz$, $C : \{z = x_0 + iy, -\infty < y < \infty\}$



$h(z) = z^2 \Rightarrow \phi(z) = x^2 - y^2$, $\psi(z) = 2xy$

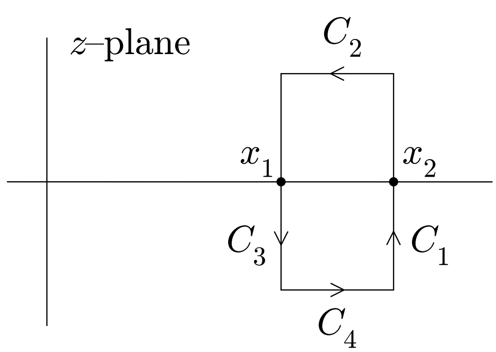
$$f(\lambda) = \int_{-\infty}^{\infty} e^{\lambda(x_0^2 - y^2 + 2i\gamma y)} i dy$$

$$= \cancelto{0}{-e^{\lambda x_0^2}\int_{-\infty}^{\infty} e^{-\lambda y^2}\sin(2\lambda x_0 y)dy} + ie^{\lambda x_0^2}\int_{-\infty}^{\infty} e^{-\lambda y^2}\cos(2\lambda x_0 y)dy$$

The integral can be evaluated directly (hw3), but here we use a different method.

claim : $f(\lambda)$ is independent of x_0

pf



$\Rightarrow \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0$: Cauchy's thm

Consider $z \in C_2$, write $z = x + iY$, $z^2 = (x^2 - Y^2) + 2ixY$

$$\left|\int_{C_2} e^{\lambda z^2} dz\right| \le \int_{C_2} \left|e^{\lambda z^2}\right| \cdot |dz| \le e^{-\lambda Y^2}\int_{x_1}^{x_2} e^{\lambda x^2} dx \to 0 \text{ as } Y \to \infty$$

same for $C_4 \Rightarrow \int_{C_1} = -\int_{C_3}$ ok

choose $x_0 = 0 \Rightarrow f(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda y^2} i dy = i\left(\frac{\pi}{\lambda}\right)^{1/2}$

note

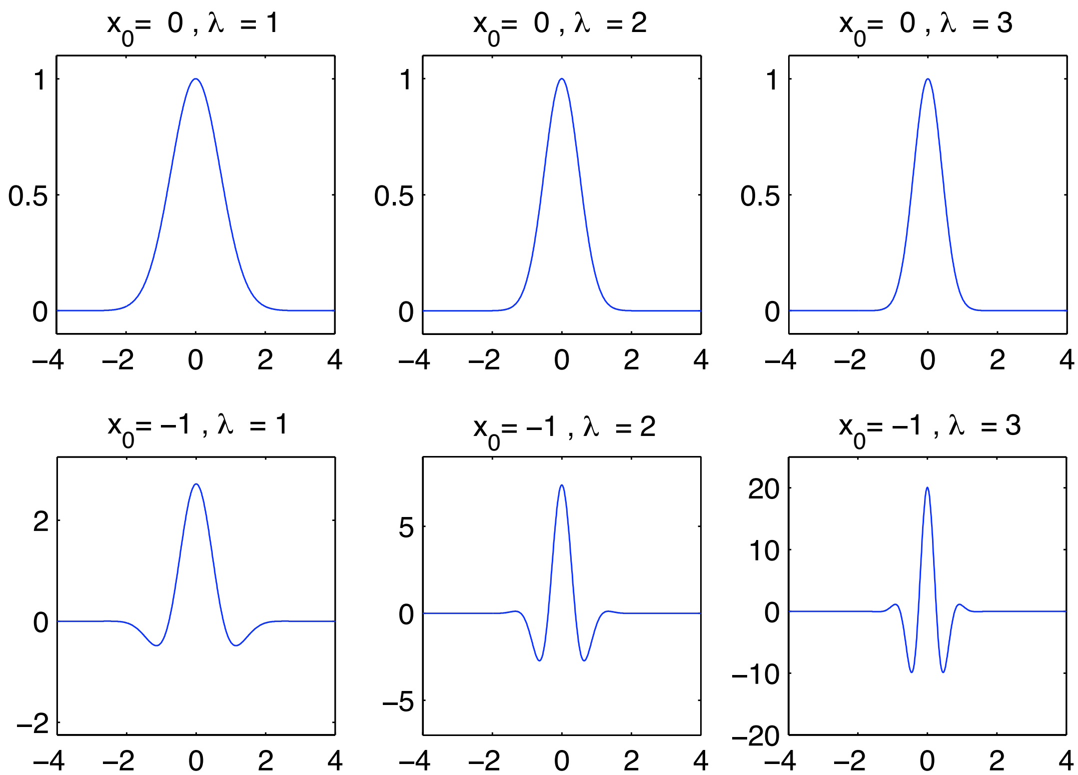
$x_0 = 0$: $\psi(x_0, y) = 0 =$ constant $\Rightarrow$ integrand is non-oscillatory

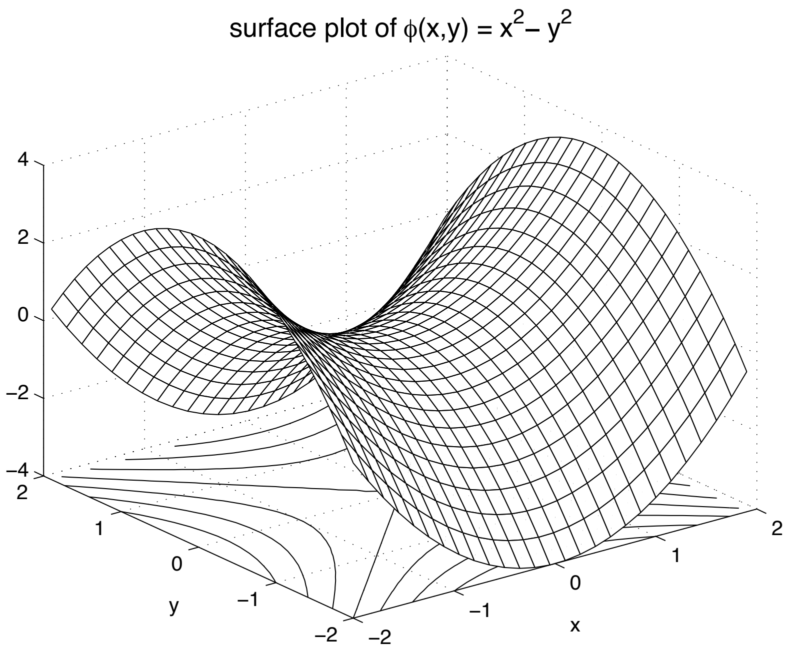
$x_0 \neq 0$: $\psi(x_0, y) \neq$ constant $\Rightarrow$ integrand is oscillatory

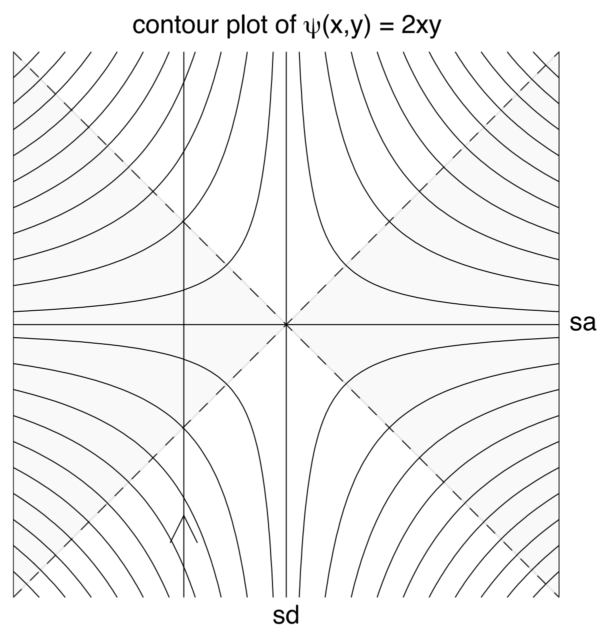
In this example the contour $\psi = 0$ has two branches; the x-axis is a steepest ascent path of ϕ (sa, $\phi > 0$, shaded) and the y-axis is a steepest descent path of ϕ (sd, $\phi < 0$, unshaded); when $x_0 = 0$, the path C coincides with sd.

$\underline{\text{ex } 0}$: $f(\lambda) = \int_C e^{\lambda z^2} dz = i\left(\frac{\pi}{\lambda}\right)^{1/2}$, $C : \{z = x_0 + iy, -\infty < y < \infty\}$

below : $\text{Re}(e^{\lambda z^2}) = e^{\lambda(x_0^2 - y^2)} \cos(2\lambda x_0 y)$, plotted as a function of y







left : $(0, 0)$ is a saddle point of $\phi(x, y)$

right : 1. shaded region : $\phi > 0$, unshaded region : $\phi < 0$

2. contours of ψ = paths of steepest ascent/descent of ϕ

<u>general case</u>

$f(\lambda) = \int_C g(z) e^{\lambda h(z)} dz \ = \int_C g(z) e^{\lambda \phi(z)} \cdot e^{i\lambda\psi(z)} dz \ , \ \lambda \to \infty$

$h(z) = h(z_0) + \cancel{h'(z_0)}(z - z_0) + \frac{1}{2} h''(z_0)(z - z_0)^2 + \cdots$: as in previous example

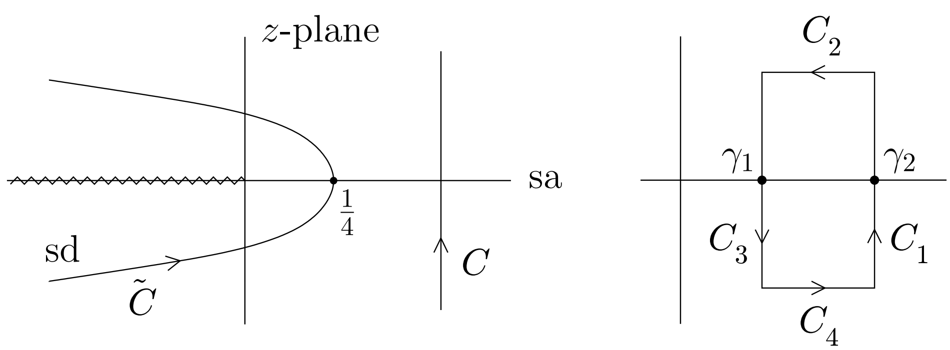
$h'(z_0) = 0$: <u>saddle point</u> , assume $h''(z_0) \neq 0$

<u>strategy</u>

1. Find the saddle point z_0.
2. Find the contours of ψ passing through z_0.
3. Determine which of these is the sd path of ϕ.
4. Deform C accordingly and apply Laplace's method.

<u>ex 1</u> : 69/2

$f(\lambda) = \int_{\gamma - i\infty}^{\gamma + i\infty} z^{-1} e^{\lambda(z - \sqrt{z})} dz \quad , \quad \gamma > 0 \quad , \quad |\arg \sqrt{z}\,| < \frac{\pi}{2} \quad , \quad \lambda \to \infty$



<u>claim</u>

The integral is (1) finite, (2) independent of γ.

<u>pf</u>

(1) $z = \gamma + iy \ \Rightarrow |\arg z| < \frac{\pi}{2} \ \Rightarrow |\arg \sqrt{z}| < \frac{\pi}{4} \ \Rightarrow \cos(\arg \sqrt{z}\,) > \frac{1}{\sqrt{2}}$

$\Rightarrow \mathrm{Re}\sqrt{z} = (\gamma^2 + y^2)^{1/4} \cos(\arg \sqrt{z}\,) > |y|^{1/2} \cdot \frac{1}{\sqrt{2}} = \sqrt{|y|/2}$

$\Rightarrow \left| \int_{\gamma - i\infty}^{\gamma + i\infty} z^{-1} e^{\lambda(z - \sqrt{z})} dz \right| \leq \int_{-\infty}^{\infty} \frac{e^{\lambda \mathrm{Re}(z - \sqrt{z})}}{(\gamma^2 + y^2)^{1/2}} dy < \int_{-\infty}^{\infty} \frac{e^{\lambda\left(\gamma - \sqrt{|y|/2}\right)}}{\gamma} dy$ <u>ok</u> 1

(2) For $z \in C_2$, write $z = x + iY$.

$\left| \int_{C_2} z^{-1} e^{\lambda(z - \sqrt{z})} dz \right| \leq \int_{\gamma_1}^{\gamma_2} \frac{e^{\lambda\left(x - \sqrt{Y/2}\right)}}{Y} dx \to 0$ as $Y \to \infty$ <u>ok</u> 2 saddle point : $h(z) = z - \sqrt{z} \Rightarrow h'(z) = 1 - \dfrac{1}{2\sqrt{z}} = 0 \;\Rightarrow\; z_0 = \frac{1}{4}$

$h''(z) = \frac{1}{4}z^{-3/2} \;\Rightarrow\; h''(z_0) = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^{-3/2} = \frac{1}{4}\cdot 8 = 2$

$h(z_0) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \;\Rightarrow\; \psi(z_0) = 0$

$z = re^{i\theta} \;\Rightarrow\; \psi(z) = \mathrm{Im}(z - \sqrt{z}\,) = r\sin\theta - \sqrt{r}\sin\frac{\theta}{2} = \sqrt{r}\sin\frac{\theta}{2}\,(2\sqrt{r}\cos\frac{\theta}{2} - 1)$

case 1 : $\sin\frac{\theta}{2} = 0 \;\Rightarrow\; \theta = \arg z = 0$: positive x-axis

$\phi(x,0) = x - \sqrt{x} = -\frac{1}{4} + (\sqrt{x} - \frac{1}{2})^2$: steepest ascent

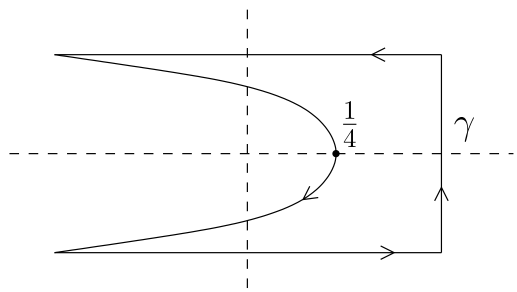
case 2 : $2\sqrt{r}\cos\frac{\theta}{2} - 1 = 0 \Rightarrow 4r\cos^2\frac{\theta}{2} = 1 \Rightarrow 4r\frac{1}{2}(\cos\theta + 1) = 1 \Rightarrow r\cos\theta + r = \frac{1}{2}$

$\sqrt{x^2+y^2} = \frac{1}{2} - x \;\Rightarrow\; \cancel{x^2} + y^2 = \frac{1}{4} - x + \cancel{x^2} \;\Rightarrow\; x = \frac{1}{4} - y^2$: parabola

$\phi(x,y) = \mathrm{Re}(z - \sqrt{z}\,) = x - \sqrt{r}\cos\frac{\theta}{2} = x - \frac{1}{2} = -\frac{1}{4} - y^2$: steepest descent

claim : $\displaystyle\int_C = \int_{\widetilde{C}}$

pf



$z = x \pm iY\,,\ -\infty < x \le \gamma \;\Rightarrow\; \mathrm{Re}\sqrt{z} > 0 \;\Rightarrow\; \left|z^{-1}e^{\lambda(z-\sqrt{z}\,)}\right| = \dfrac{e^{\lambda\mathrm{Re}(z-\sqrt{z}\,)}}{(x^2+Y^2)^{1/2}} \le \dfrac{e^{\lambda x}}{|Y|}$

$\left|\displaystyle\int_{-\infty+iY}^{\gamma+iY} z^{-1}e^{\lambda(z-\sqrt{z}\,)}dz\right| \le \displaystyle\int_{-\infty}^{\gamma}\dfrac{e^{\lambda x}}{|Y|}dx = \dfrac{e^{\lambda\gamma}}{\lambda|Y|} \to 0$ as $|Y| \to \infty$ ok

for $z \in \widetilde{C}$ set $h(z) = h(z_0) - s^2 \Rightarrow z - \sqrt{z} = -\frac{1}{4} - s^2$

$h(z) = \cancel{h(z_0)} + \cancel{h'(z_0)}(z - z_0) + \frac{1}{2}h''(z_0)(z-z_0)^2 + O((z-z_0)^3) = \cancel{h(z_0)} - s^2$

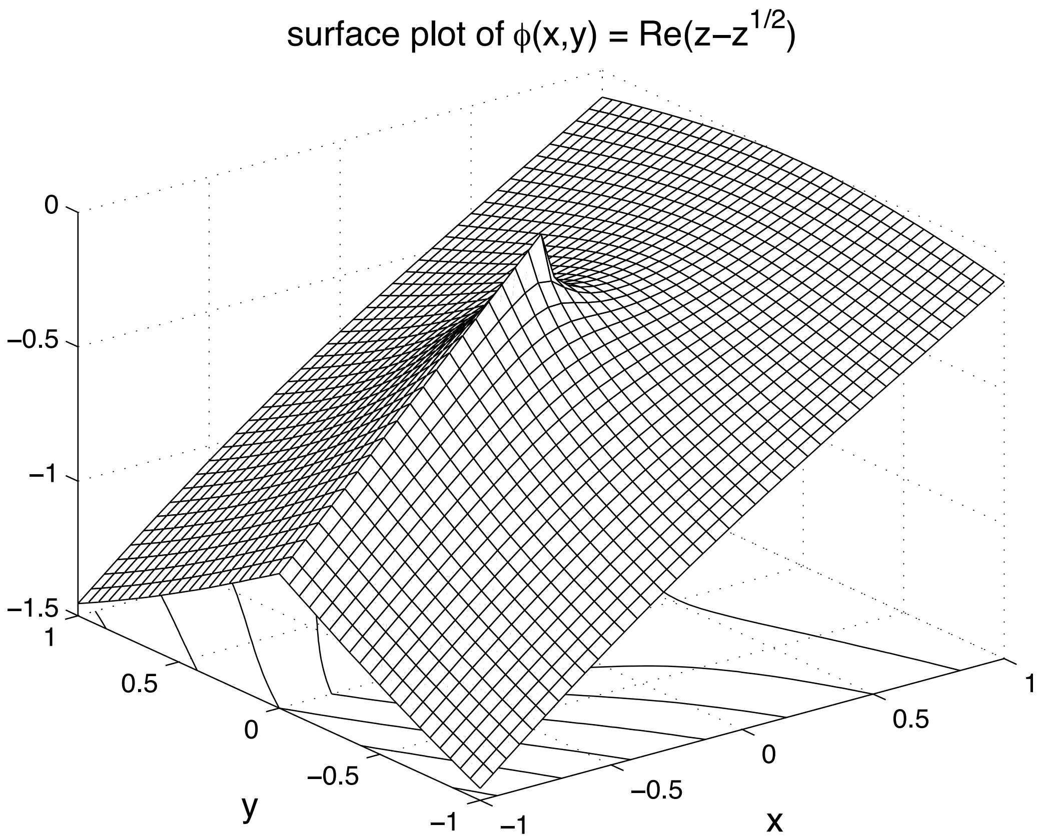
$(z - \frac{1}{4})^2 = -s^2 + O(s^3) \;\Rightarrow\; z = \frac{1}{4} \pm is + O(s^2)$, choose +

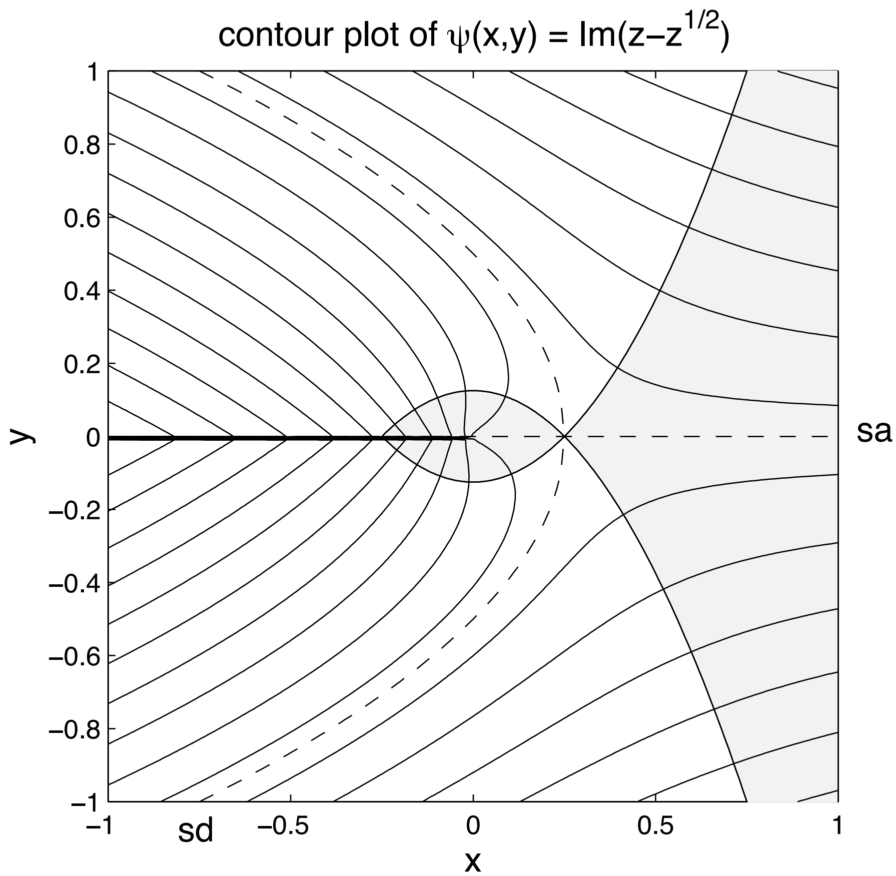
$\displaystyle\int_{\widetilde{C}} z^{-1}e^{\lambda(z-\sqrt{z}\,)}dz \sim \int_{-\infty}^{\infty}\frac{e^{-\lambda(\frac{1}{4}+s^2)}}{z(0)}z'(0)ds \;=\; e^{-\lambda/4}\int_{-\infty}^{\infty}\frac{e^{-\lambda s^2}}{1/4}ids$

$f(\lambda) = \displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} z^{-1}e^{\lambda(z-\sqrt{z}\,)}dz \;\sim\; 4ie^{-\lambda/4}\left(\frac{\pi}{\lambda}\right)^{1/2}$ as $\lambda \to \infty$

note : sd path of $\phi \Rightarrow$ direction of $-\nabla\phi \Rightarrow$ orthogonal to $\nabla\psi \Rightarrow$ contour of ψ

pf : Cauchy-Riemann eqs : $\psi_x = \psi_y, \phi_y = -\psi_x \Rightarrow \nabla\phi\cdot\nabla\psi = 0$ ok


surface plot of φ(x,y) = Re(z−z^{1/2})
0
−0.5
−1
−1.5
1
0.5
0
−0.5
−1
y
−1
−0.5
0
0.5
1
x


contour plot of ψ(x,y) = Im(z−z^{1/2})
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
y
sa
−1
sd
−0.5
0
0.5
1
x

ex 2

$w''(\lambda) - \lambda w(\lambda) = 0$: Airy equation , "of fundamental importance"

1. two linearly independent solutions : $\mathrm{Ai}(\lambda)$, $\mathrm{Bi}(\lambda)$
2. $\lambda > 0$: exponential growth/decay , $\lambda < 0$: oscillatory

motivation

$u_t + u_{xxx} = 0$, $u(x,t) = t^\alpha f(xt^\beta)$: similarity solution

$$\cancel{t^\alpha} f' x\beta t^{\beta-1} + \alpha \cancel{t^{\alpha-1}} f + \cancel{t^\alpha} f''' t^{3\beta} = 0$$

$$\beta x t^\beta \cancel{t^{-1}} f' + \alpha \cancel{t^{-1}} f + \cancel{t^{3\beta}} f''' = 0 \ , \ \text{choose } 3\beta = -1 \Rightarrow \beta = -\tfrac{1}{3}$$

$$\beta x t^\beta f' + \alpha f + f''' = 0 \ , \ \text{set } s = xt^\beta \, , \, \alpha = \beta$$

$$f''' - \tfrac{1}{3}(sf' + f) = 0 \Rightarrow 3f''' - (sf)' = 0$$

$$\Rightarrow \ 3f'' - sf = 0 \text{ (for example)}$$

set $r = cs$, $g(r) = f(s) \Rightarrow g'(r) = f'(s)c^{-1}$, $g''(r) = f''(s)c^{-2}$

$$\Rightarrow \ 3c^2 g'' - c^{-1} r g = 0 \ , \ \text{choose } 3c^3 = 1 \Rightarrow c = 3^{-1/3}$$

$\Rightarrow \ g''(r) - rg(r) = 0 \Rightarrow u(x,t) = t^{-1/3}\mathrm{Ai}(x(3t)^{-1/3})$ is a solution of the PDE

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return to $w''(\lambda) - \lambda w(\lambda) = 0$

$w(\lambda) = \frac{1}{2\pi i}\int_C g(z)e^{\lambda z}dz$: general transform

$$\begin{aligned} w''(\lambda) - \lambda w(\lambda) &= \tfrac{1}{2\pi i}\int_C z^2 g(z)e^{\lambda z}dz \ - \ \tfrac{1}{2\pi i}\int_C g(z)\lambda e^{\lambda z}dz \\ &= \tfrac{1}{2\pi i}\int_C z^2 g(z)e^{\lambda z}dz \ - \ \tfrac{1}{2\pi i}\Big(g(z)e^{\lambda z}\Big|_a^b - \int_C g'(z)e^{\lambda z}dz\Big) \\ &= \tfrac{1}{2\pi i}\int_C (z^2 g(z) + g'(z))e^{\lambda z}dz \ - \ \tfrac{1}{2\pi i}\, g(z)e^{\lambda z}\Big|_a^b \end{aligned}$$

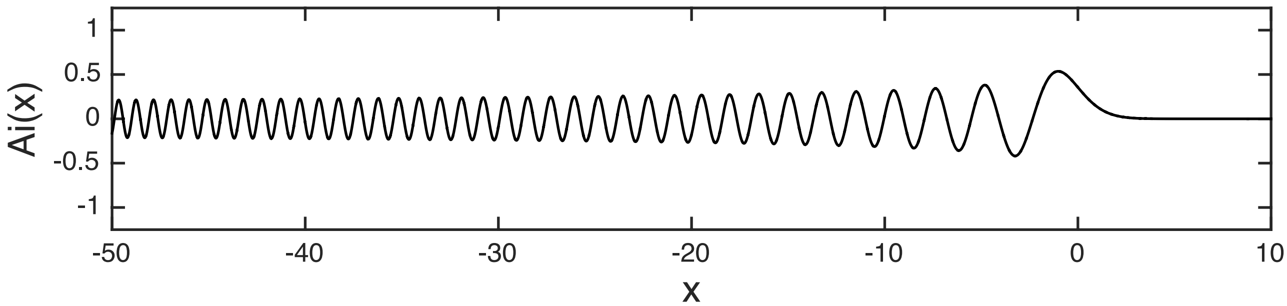
hence we require $z^2 g(z) + g'(z) = 0$, $\lim\limits_{z\to a,b} g(z)e^{\lambda z} = 0$

$$g' + z^2 g = 0 \ \Rightarrow \ g' = -z^2 g \ \Rightarrow \ \frac{dg}{g} = -z^2 dz \Rightarrow \ \log g = -\tfrac{1}{3}z^3 \ \Rightarrow \ \ g = e^{-\frac{1}{3}z^3}$$

$$z = re^{i\theta} \ \Rightarrow \ |g(z)e^{\lambda z}| = |e^{\lambda z - \frac{1}{3}z^3}| = e^{\lambda r\cos\theta - \frac{1}{3}r^3\cos 3\theta}$$

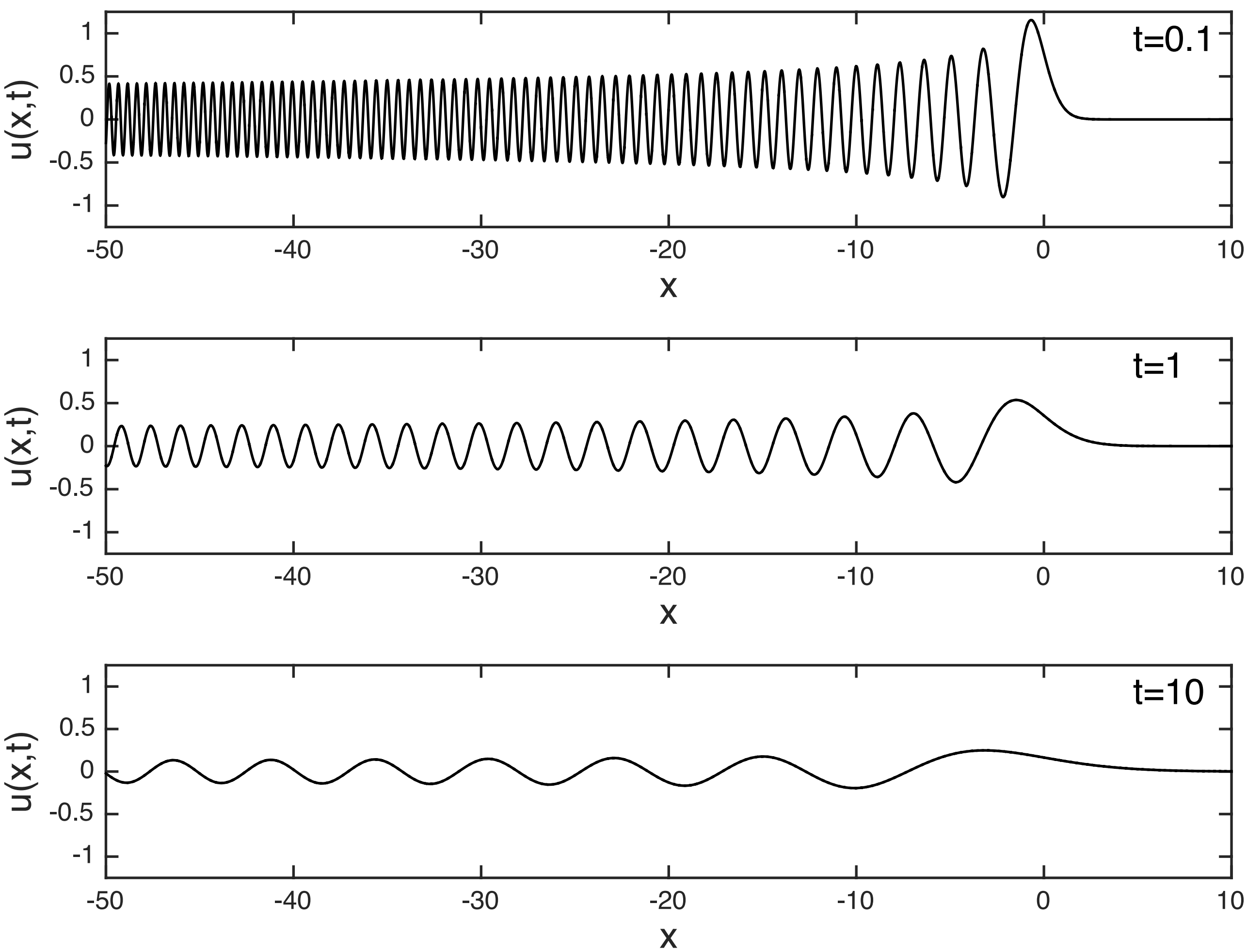
ok if $r \to \infty$ and $\cos 3\theta = 1 \ \Rightarrow \ 3\theta = 0\, , \, 2\pi\, , \, 4\pi \ \Rightarrow \ \theta = 0\, , \, \frac{2\pi}{3}\, , \, \frac{4\pi}{3}$

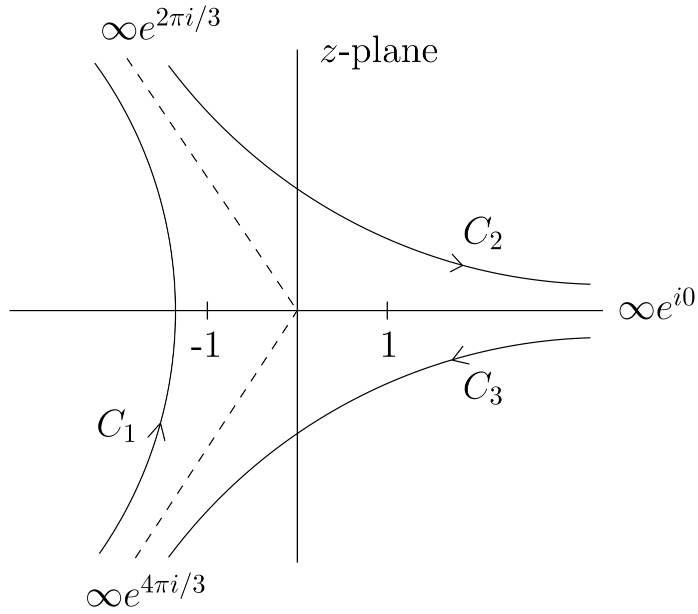
$w''(\lambda) - \lambda w(\lambda) = 0$: Airy equation



$u_t + u_{xxx} = 0$: linear dispersive PDE

$u(x,t) = t^{-1/3}\mathrm{Ai}(x(3t)^{-1/3})$: similarity solution of the PDE





define C_j , $I_j(\lambda) = \frac{1}{2\pi i}\int_{C_j} e^{\lambda z - \frac{1}{3}z^3} dz$: solutions of Airy equation

note : $I_1 + I_2 + I_3 = 0 \Rightarrow$ the $I_j(\lambda)$ are linearly dependent

$\mathrm{Ai}(\lambda) = \frac{1}{2\pi i}\int_{C_1} e^{\lambda z - \frac{1}{3}z^3} dz$, $\lambda \to \infty$

$h(z) = z$ has no saddle point , set $\lambda = \nu^2$, $\nu > 0$ and $z = \nu\zeta$

$\mathrm{Ai}(\nu^2) = \frac{1}{2\pi i}\int_{C_1} e^{\nu^3(\zeta - \frac{1}{3}\zeta^3)} \cdot \nu d\zeta = \frac{\nu}{2\pi i}\int_{C_1} e^{\nu^3(z - \frac{1}{3}z^3)} dz$

$h(z) = z - \frac{1}{3}z^3 \Rightarrow h'(z) = 1 - z^2 \Rightarrow z_0 = \pm 1$: 2 saddle points , $h''(z_0) = -2z_0$

$\underline{z_0 = -1}$: find sa, sd paths of ϕ through z_0

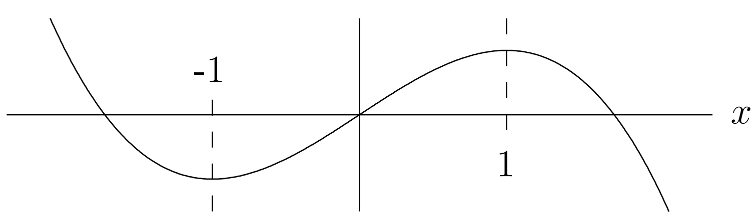
$h(-1) = -1 - \frac{1}{3}(-1) = -\frac{2}{3} \Rightarrow \psi(-1, 0) = 0$

$z = x + iy \Rightarrow h(z) = x + iy - \frac{1}{3}(x^3 + 3ix^2y - 3xy^2 - iy^3)$

$\psi(x, y) = y - \frac{1}{3}(3x^2y - y^3) = \frac{1}{3}y(y^2 - 3x^2 + 3)$

<u>case 1</u>

$y = 0$: real axis $\Rightarrow \phi(x, 0) = x - \frac{1}{3}x^3 \Rightarrow \phi(-1, 0) = -\frac{2}{3}$, $\phi(1, 0) = \frac{2}{3}$



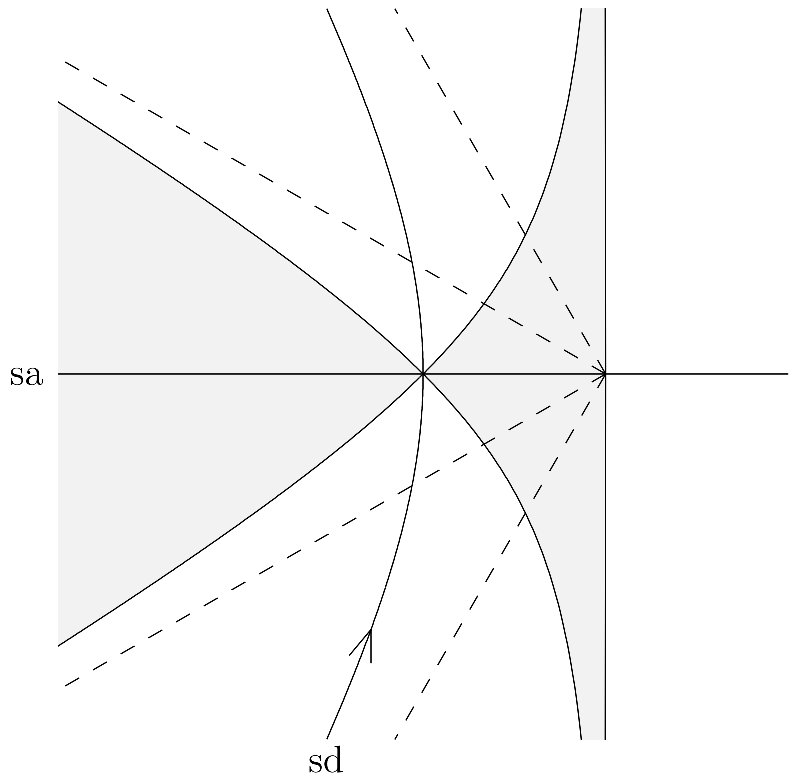
local min at $x = -1 \Rightarrow$ real axis is the sa path of ϕ through $z_0 = -1$

<u>case 2</u>

$y^2 - 3x^2 + 3 = 0$: hyperbola , $x = \pm(1 + \frac{1}{3}y^2)^{1/2}$

choose branch passing through $(-1, 0)$

asymptotes : $y = \pm\sqrt{3}x \;\Rightarrow\; \theta = \frac{2\pi}{3}\,,\, \frac{4\pi}{3}$



shaded : $\phi > \phi_0$

unshaded : $\phi < \phi_0$

$\phi(x,y) = x - \frac{1}{3}(x^3 - 3xy^2) = \frac{1}{3}x(3y^2 - x^2 + 3)$: evaluate on hyperbola

$x = -(1 + \frac{1}{3}y^2)^{1/2} = -(1 + \frac{1}{6}y^2 + O(y^4))$

$\phi(x,y) = -\frac{1}{3}(1 + \frac{1}{6}y^2 + O(y^4))(3y^2 - (1 + \frac{1}{3}y^2) + 3) = -\frac{2}{3} - y^2 + O(y^4)$

$\Rightarrow$ hyperbola is the sd path of ϕ through $z_0 = -1$

find curves where $\phi = \phi_0 = \phi(-1, 0) = -\frac{2}{3}$

$\phi(x,y) = x - \frac{1}{3}x^3 + xy^2 = -\frac{2}{3} \;\Rightarrow\; 3xy^2 - x^3 + 3x + 2 = 0$: what shape is this?

$x \to -\infty \Rightarrow 3xy^2 - x^3 + \cancel{3x} + \cancel{2} = 0 \Rightarrow 3y^2 \sim x^2 \Rightarrow y \sim \pm\frac{1}{\sqrt{3}}x$: $\theta = \frac{5\pi}{6}\,,\, \frac{7\pi}{6}$

$x \to 0^- \Rightarrow 3xy^2 - \cancel{x^3} + \cancel{3x} + 2 = 0 \Rightarrow y^2 \sim -\frac{2}{3x} \Rightarrow y \sim \pm\sqrt{-\frac{2}{3x}}$

deform C_1 into sd , Laplace's method : $h(z) = h(z_0) - s^2$

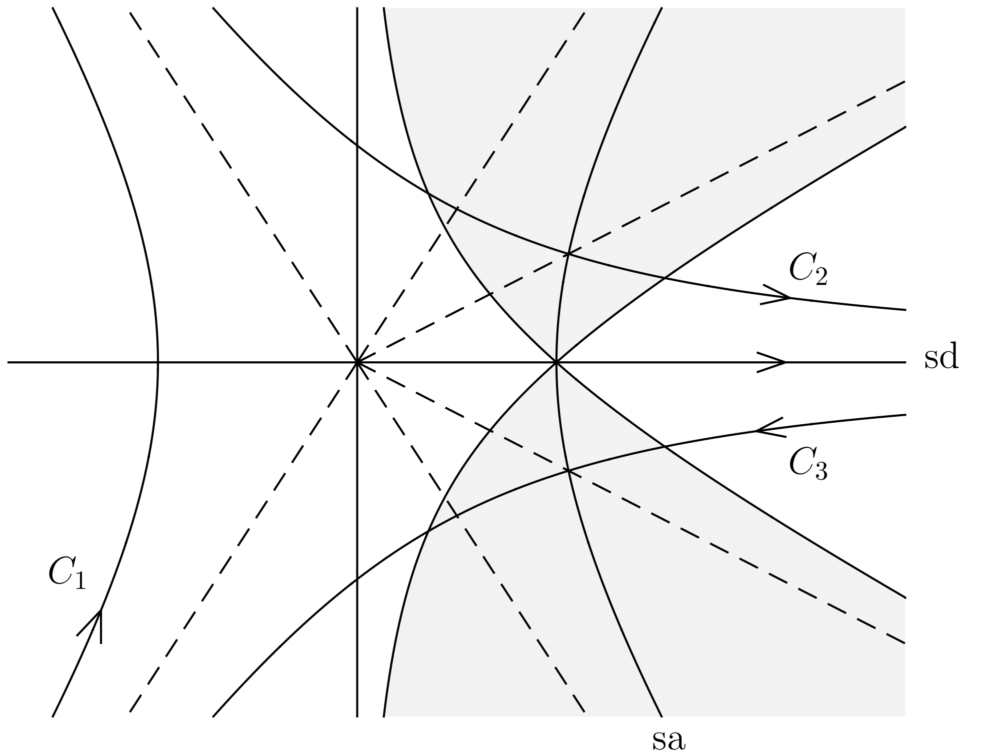
$\cancel{h(z_0)} + \cancel{h'}(z_0)(z - z_0) + \frac{1}{2}h''(z_0)(z - z_0)^2 + O((z - z_0)^3) = \cancel{h(z_0)} - s^2$

$(z + 1)^2 = -s^2 + O(s^3) \;\Rightarrow\; z = -1 \pm is + O(s^2)$, choose $+$

$$\mathrm{Ai}(\nu^2) \sim \frac{\nu}{2\pi i}\int_{-\infty}^{\infty} e^{-\nu^3(\frac{2}{3}+s^2)} i ds \;=\; \frac{\nu}{2\pi} e^{-\frac{2}{3}\nu^3}\left(\frac{\pi}{\nu^3}\right)^{1/2}$$

$\mathrm{Ai}(\lambda) \sim \frac{1}{2\sqrt{\pi}}\lambda^{-\frac{1}{4}}e^{-\frac{2}{3}\lambda^{3/2}}$ as $\lambda \to \infty$

$\underline{z_0 = 1}$



$h(1) = 1 - \frac{1}{3}(1) = \frac{2}{3} \;\Rightarrow\; \psi(1,0) = 0$

$\psi(x,y) = \frac{1}{3}y(y^2 - 3x^2 + 3) = 0$: same as before

case 1 : $y = 0$: real axis , $\phi(x,0) = x - \frac{1}{3}x^3$: local max at $x = 1$

$\Rightarrow$ real axis is the sd path of ϕ through $z_0 = 1$

case 2 : $y^2 - 3x^2 + 3 = 0$: hyperbola , asymptotes : $y = \pm\sqrt{3}x$, $\theta = \pm\frac{\pi}{3}$

choose branch passing through $z_0 = 1$

$\phi(x,y) = \cdots = \frac{2}{3} + y^2 + O(y^4)$ on hyperbola near $z_0 = 1$

$\Rightarrow$ hyperbola is the sa path of ϕ through $z_0 = 1$

$\phi = \phi_0 = \phi(1,0) = \frac{2}{3} \Rightarrow \cdots \Rightarrow y \sim \pm\frac{1}{\sqrt{3}}x$ for $x \to \infty$, $y \sim \pm\sqrt{-\frac{2}{3x}}$ for $x \to 0^+$

C_1 cannot be deformed into sd

C_2 , C_3 can be deformed into sd for $x > 0$ plus a ray for $x < 0$

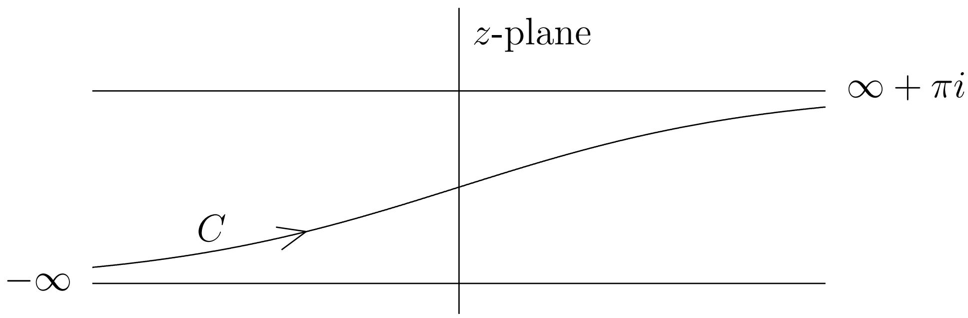
def: $\mathrm{Bi}(\lambda) = \frac{1}{2\pi}\Big(\int_{C_2} - \int_{C_3}\Big)e^{\lambda z - \frac{1}{3}z^3}dz \;\sim\; \frac{1}{\sqrt{\pi}}\lambda^{-1/4}e^{\frac{2}{3}\lambda^{3/2}}$ as $\lambda \to \infty$: hw3

$\mathrm{Ai}(-\lambda) \sim \frac{1}{\sqrt{\pi}}\lambda^{-1/4}\sin(\frac{2}{3}\lambda^{3/2} + \frac{\pi}{4})$, $\mathrm{Bi}(-\lambda) \sim \frac{1}{\sqrt{\pi}}\lambda^{-1/4}\cos(\frac{2}{3}\lambda^{3/2} + \frac{\pi}{4})$ as $\lambda \to \infty$

<u>ex 3</u>

$H_s^{(1)}(\lambda) = \frac{1}{\pi i}\int_C e^{\lambda \sinh z - sz} dz$: <u>Hankel function</u>

$s > 0$, C goes from $-\infty$ to $\infty + \pi i$



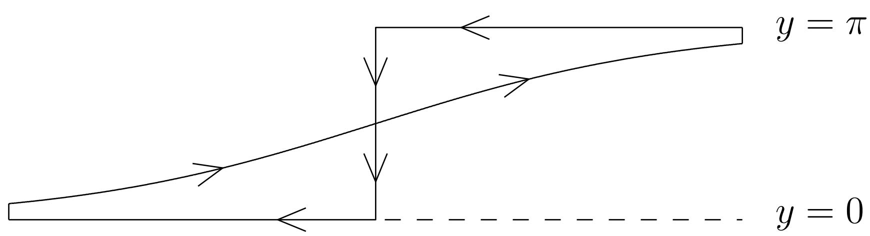
<u>claim</u> : The integral is finite.

<u>pf</u>

$\sinh z = \sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$

$\cosh z = \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$

$\text{Re}(\lambda \sinh z - sz) = \lambda \sinh x \cos y - sx$



1. $y = 0 \Rightarrow \text{Re}(\lambda \sinh z - sz) = \lambda \sinh x - sx < 0$ for $x < 0$ and λ large

$\left|\int_{-\infty}^{0} e^{\lambda \sinh z - sz} dz\right| \le \int_{-\infty}^{0} e^{\lambda \sinh x - sx} dx$: finite

2. $y = \pi \Rightarrow \text{Re}(\lambda \sinh z - sz) = -\lambda \sinh x - sx < 0$ for $x > 0$

$\left|\int_{\pi i}^{\infty+\pi i} e^{\lambda \sinh z - sz} dz\right| \le \int_{0}^{\infty} e^{-(\lambda \sinh x + sx)} dx$: finite

3. $x = X \Rightarrow \text{Re}(\lambda \sinh z - sz) = \lambda \sinh X \cos y - sX < 0$ for $y \sim \pi$

$\left|\int_{X+\pi i-\epsilon i}^{X+\pi i} e^{\lambda \sinh z - sz} dz\right| \le \int_{\pi-\epsilon}^{\pi} e^{\lambda \sinh X \cos y - sX} dy \to 0$ as $X \to \infty$, $\epsilon \to 0$

4. $x = -X \Rightarrow \text{Re}(\lambda \sinh z - sz) = -\lambda \sinh X \cos y - sX < 0$ for $y \sim 0$

$\left|\int_{-X}^{-X+\epsilon i} e^{\lambda \sinh z - sz} dz\right| \le \int_{0}^{\epsilon} e^{-\lambda \sinh X \cos y - sX} dy \to 0$ as $X \to \infty$, $\epsilon \to 0$ <u>ok</u>

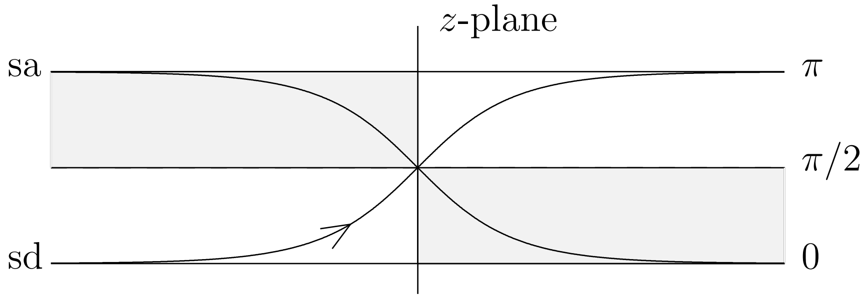
$$\int_C e^{\lambda \sinh z - sz} dz = \int_C e^{\lambda h(z)} g(z) dz \ , \ g(z) = e^{-sz} \ , \ h(z) = \sinh z$$

$h'(z) = \cosh z = 0 \ \Rightarrow \ z_0 = (n + \frac{1}{2})\pi i \, , \, n = 0, \pm 1, \pm 2, \ldots \ , \ h''(z_0) = i(-1)^n$

consider $n = 0 \ , \ z_0 = \frac{\pi i}{2} \ , \ h(z_0) = i \ \Rightarrow \ \psi(z_0) = 1$: find the contours

$\psi(x, y) = \mathrm{Im} h(z) = \cosh x \sin y = 1 \ \Rightarrow \ \sin y = \frac{1}{\cosh x}$: one point is $(0, \frac{\pi}{2})$

For any $x \neq 0$, there are 2 values of y : $0 < y_1 < \frac{\pi}{2} < y_2 < \pi$.



$\phi(x, y) = \mathrm{Re}\, h(z) = \sinh x \cos y$

$\phi(0, \frac{\pi}{2}) = 0 \ , \ \phi(x, y) = 0 \Rightarrow x = 0$ or $y = \frac{\pi}{2}$

$x < 0 \, , \, 0 < y < \frac{\pi}{2} \ \Rightarrow \ \phi < 0$

$x < 0 \, , \, \frac{\pi}{2} < y < \pi \ \Rightarrow \ \phi > 0$

$x > 0 \, , \, 0 < y < \frac{\pi}{2} \ \Rightarrow \ \phi > 0$

$x > 0 \, , \, \frac{\pi}{2} < y < \pi \ \Rightarrow \ \phi < 0$

Deform C into the sd path of ϕ through $z_0 = \frac{\pi i}{2}$.

Laplace's method : $h(z) = h(z_0) - t^2$

$\cancel{h(z_0)} + \cancel{h'(z_0)}(z - z_0) + \frac{1}{2} h''(z_0)(z - z_0)^2 + O((z - z_0)^3) = \cancel{h(z_0)} - t^2$

$\frac{i}{2}(z - \frac{\pi i}{2})^2 = -t^2 + O(t^3) \ \Rightarrow \ (z - \frac{\pi i}{2})^2 = 2it^2 + O(t^3)$

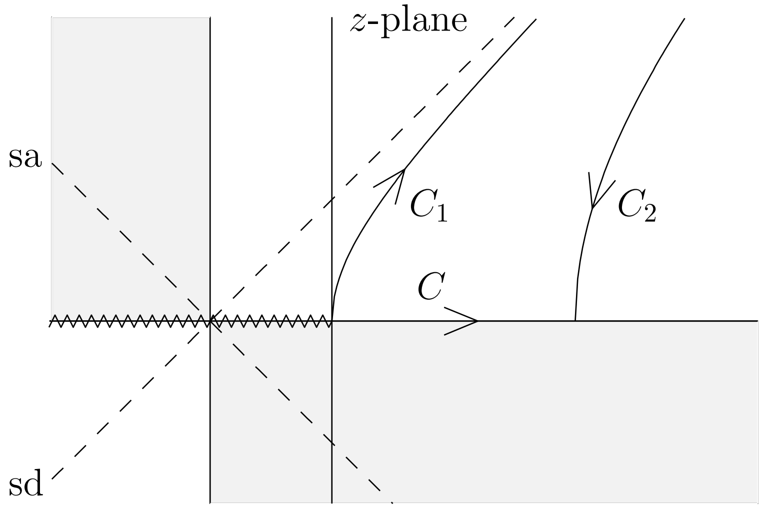
$\Rightarrow \ z = \frac{\pi i}{2} \pm \sqrt{2} e^{\pi i/4} t + O(t^2)$, choose +

$$H_s^{(1)}(\lambda) = \frac{1}{\pi i} \int_{\mathrm{sd}} e^{\lambda \sinh z - sz} dz \ \sim \ \frac{1}{\pi i} \int_{-\infty}^{\infty} e^{\lambda(i - t^2) - sz(0)} z'(0) dt$$

$$= \frac{e^{i\lambda}}{\pi i} \int_{-\infty}^{\infty} e^{-\lambda t^2} \cdot e^{-s\pi i/2} \cdot \sqrt{2} e^{\pi i/4} dt \ = \frac{e^{i\lambda}}{\pi i} \cdot e^{-s\pi i/2} \cdot e^{\pi i/4} \sqrt{2} \cdot \left(\frac{\pi}{\lambda}\right)^{1/2}$$

$$H_s^{(1)}(\lambda) \sim e^{i(\lambda - s\pi/2 - \pi/4)} \cdot \left(\frac{2}{\pi \lambda}\right)^{1/2} \text{ as } \lambda \to \infty$$

ex 4 : $f(\lambda) = \int_0^1 \frac{e^{i\lambda(z+z^2)}}{\sqrt{z}} dz$, $-\frac{\pi}{2} < \arg\sqrt{z} < \frac{\pi}{2}$, integral is finite



$h(z) = i(z+z^2) \Rightarrow h'(z) = i(1+2z) = 0 \Rightarrow z_0 = -\frac{1}{2}$, $h''(z_0) = 2i$

$h(-\frac{1}{2}) = i(-\frac{1}{2}+\frac{1}{4}) = -\frac{i}{4} \Rightarrow \psi(z_0) = -\frac{1}{4}$

$h(z) = i(x+iy+x^2-y^2+2ixy) = -y(1+2x) + i(x^2-y^2+x)$

$\psi(x,y) = x^2-y^2+x = -\frac{1}{4} \Rightarrow (x+\frac{1}{2})^2 - y^2 = 0 \Rightarrow y = \pm(x+\frac{1}{2})$

case 1 : $y = x+\frac{1}{2}$, $\phi(x,y) = -y(1+2x) = -2y^2$: sd

case 2 : $y = -(x+\frac{1}{2})$, $\phi(x,y) = -y(1+2x) = 2y^2$: sa

note : $\phi(z_0) = 0 \Rightarrow -y(1+2x) = 0 \Rightarrow y = 0$ or $x = -\frac{1}{2}$

C cannot be deformed into sd path through $z_0 = -\frac{1}{2}$; consider instead sd paths through $z = 0, 1$

$z = 0$: $h(0) = 0 \Rightarrow \psi(0) = 0$

$\psi(x,y) = x^2-y^2+x = (x+\frac{1}{2})^2 - y^2 - \frac{1}{4} = 0$

hyperbola , asymptotes : $y = \pm(x+\frac{1}{2})$

Let C_1 denote the upper part of the right branch; this is a sd path of ϕ through $z = 0$ which is asymptotic to the sd path of ϕ through $z_0 = -\frac{1}{2}$.

$z = 1$: $h(1) = 2i \Rightarrow \psi(1) = 2$

$\psi(x,y) = x^2-y^2+x = (x+\frac{1}{2})^2 - y^2 - \frac{1}{4} = 2$: hyperbola , same asymptotes

Let C_2 ...; deform C into $C_1 + C_2$; check integral is finite; apply Laplace's method; note that $\psi|_{C_1,C_2}$ is constant and $\phi|_{C_1,C_2}$ has a maximum at $y = 0$.

$$f(\lambda) = \Big(\int_{C_1} + \int_{C_2}\Big)\frac{e^{i\lambda(z+z^2)}}{\sqrt{z}}dz$$

<u>on C_1</u> : $h(z) = h(0) - s \;\Rightarrow\; \cancel{h(0)} + h'(0)z + O(z^2) = \cancel{h(0)} - s$

$\Rightarrow\; iz = -s + O(s^2) \;\Rightarrow\; z = is + O(s^2) \;\Rightarrow\; \sqrt{z} = e^{\pi i/4}s^{1/2} + O(s^{3/2})$

$$\int_{C_1} \frac{e^{i\lambda(z+z^2)}}{\sqrt{z}}dz \;\sim\; \int_0^\infty e^{-\lambda s}\cdot e^{-\pi i/4}s^{-1/2}\cdot i ds \;=\; e^{\pi i/4}\frac{\Gamma(\frac{1}{2})}{\lambda^{1/2}} \text{ as } \lambda \to \infty$$

<u>on C_2</u> : $h(z) = h(1) - s \;\Rightarrow\; \cancel{h(1)} + h'(1)(z-1) + O((z-1)^2) = \cancel{h(1)} - s$

$\Rightarrow\; 3i(z-1) = -s + O(s^2) \;\Rightarrow\; z = 1 + \frac{1}{3}is + O(s^2) \;\Rightarrow\; \sqrt{z} = 1 + O(s)$

$$\int_{C_2} \frac{e^{i\lambda(z+z^2)}}{\sqrt{z}}dz \;\sim\; -\int_0^\infty e^{\lambda(2i-s)}\cdot \tfrac{1}{3}ids \;=\; -\frac{ie^{2\lambda i}}{3\lambda} \text{ as } \lambda \to \infty$$

$$\int_0^1 \frac{e^{i\lambda(z+z^2)}}{\sqrt{z}}dz \;\sim\; \Big(\frac{\pi}{\lambda}\Big)^{1/2}e^{\pi i/4} - \frac{ie^{2\lambda i}}{3\lambda} \text{ as } \lambda \to \infty$$