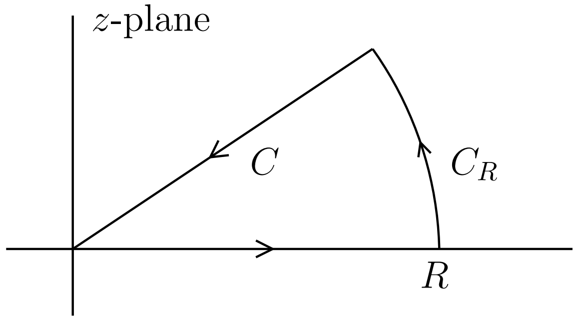
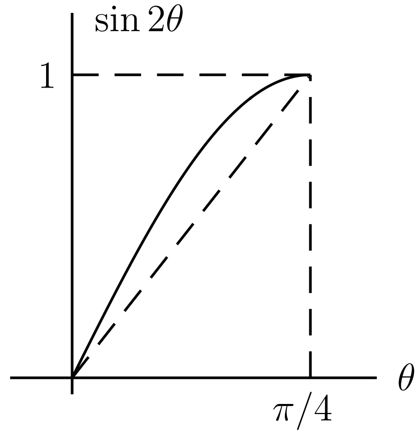
4.1 method of stationary phase

$f(\lambda) = \int_a^b g(t) e^{i\lambda h(t)} dt$, $\lambda \to \infty$, assume t , $g(t)$, $h(t)$ are real

ex : $\int_{-\infty}^{\infty} e^{\pm i\lambda t^2} dt = \left(\frac{\pi}{\lambda}\right)^{1/2} e^{\pm \pi i/4}$ for $\lambda > 0$, recall : $\int_{-\infty}^{\infty} e^{-\lambda t^2} dt = \left(\frac{\pi}{\lambda}\right)^{1/2}$

pf : consider + case , − case follows similarly





$$\left(\int_0^R + \cancel{\int_{C_R}} + \int_C \right) e^{i\lambda z^2} dz = 0$$

1. $z \in C \Rightarrow z = re^{\pi i/4} \Rightarrow i\lambda z^2 = -\lambda r^2$

$$\lim_{R\to\infty} \int_C e^{i\lambda z^2} dz = -\int_0^{\infty} e^{-\lambda r^2} e^{\pi i/4} dr = -\frac{1}{2}\left(\frac{\pi}{\lambda}\right)^{1/2} e^{\pi i/4}$$

2. $z \in C_R \Rightarrow z = Re^{i\theta} \Rightarrow i\lambda z^2 = i\lambda R^2(\cos 2\theta + i \sin 2\theta)$

$$\left| \int_{C_R} e^{i\lambda z^2} dz \right| \le \int_0^{\pi/4} e^{-\lambda R^2 \sin 2\theta} R\, d\theta \le R \int_0^{\pi/4} e^{-\lambda R^2 \cdot 4\theta/\pi} d\theta$$

note : $0 \le \theta \le \pi/4 \Rightarrow \sin 2\theta \ge 4\theta/\pi$

$$= Re^{-\lambda R^2 \cdot 4\theta/\pi} \cdot \frac{\pi}{-4\lambda R^2}\Big|_0^{\pi/4} = \frac{\pi}{4\lambda R}(1 - e^{-\lambda R^2}) \to 0 \text{ as } R \to \infty \quad \underline{\text{ok}}$$

$f(\lambda) = \int_a^b g(t) e^{i\lambda h(t)} dt$, $\lambda \to \infty$, assume t , $g(t)$, $h(t)$ are real

$e^{i\lambda h(t)} = \cos \lambda h(t) + i \sin \lambda h(t)$, $\lambda h(t)$: phase , integrand oscillates in sign

$h(t) = h(t_0) + h'(t_0)(t - t_0) + \frac{1}{2} h''(t_0)(t - t_0)^2 + \cdots$

$e^{i\lambda h(t)} = e^{i\lambda h(t_0)} \cdot e^{i\lambda h'(t_0)(t-t_0)} \cdot e^{i\lambda \frac{1}{2} h''(t_0)(t-t_0)^2} \cdots$

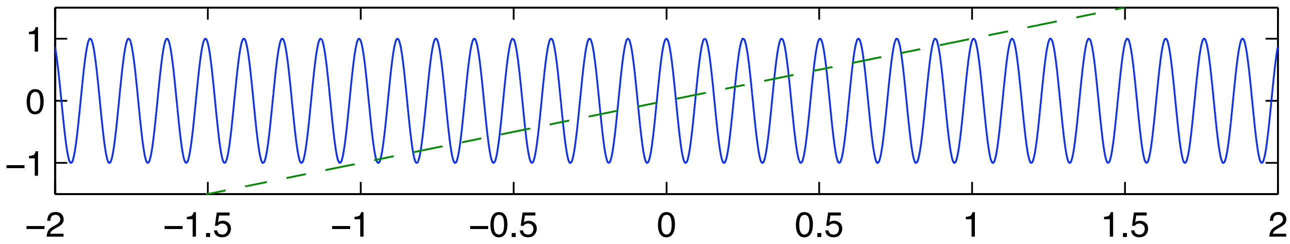
$h'(t_0) \ne 0 \Rightarrow$ phase varies relatively rapidly for $t \approx t_0$

$h'(t_0) = 0 \Rightarrow$ phase varies relatively slowly for $t \approx t_0$

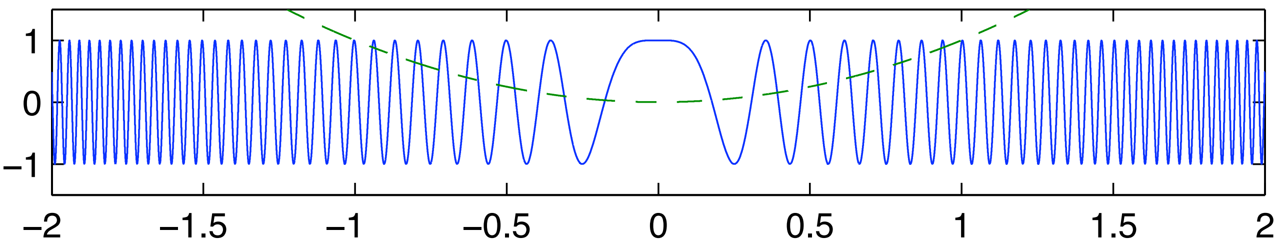
def: If $h'(t_0) = 0$, then t_0 is a point of stationary phase; near such points the oscillatory contributions to the integral cancel less strongly than near points where $h'(t_0) \ne 0$.

These are plots of $\cos \lambda h(t)$ for $\lambda = 50$ and various $h(t)$.

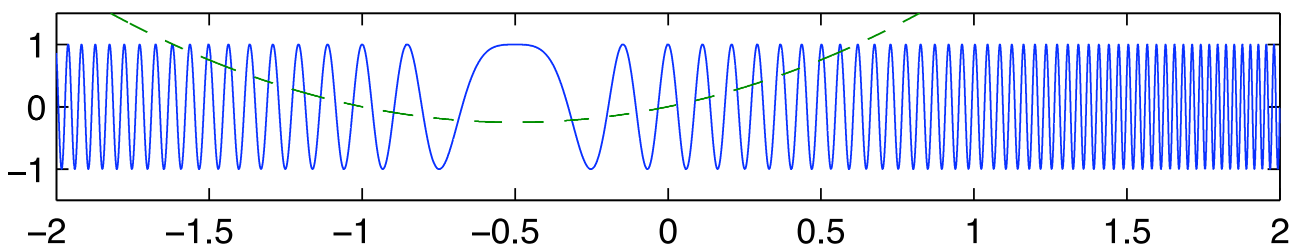
$h(t) = t$: no point of stationary phase



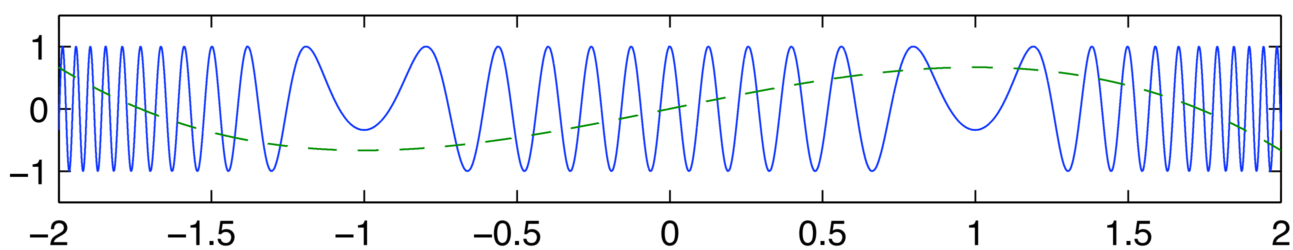
$h(t) = t^2$, $t_0 = 0$



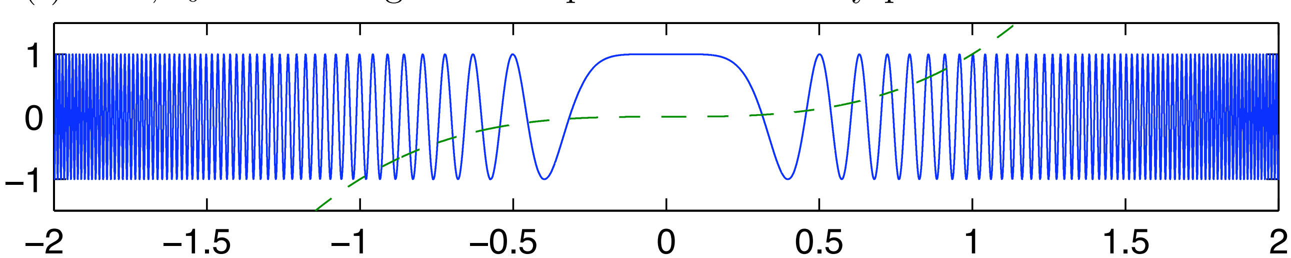
$h(t) = t + t^2$, $t_0 = -\frac{1}{2}$



$h(t) = t - \frac{1}{3}t^3$, $t_0 = \pm 1$: two points of stationary phase



$h(t) = t^3$, $t_0 = 0$: a higher order point of stationary phase



<u>case 1</u> : $h'(t) \neq 0$ for all $t \in [a,b]$

$$f(\lambda) = \int_a^b \frac{g(t)}{i\lambda h'(t)}\, i\lambda h'(t) e^{i\lambda h(t)} dt = \frac{g(t)}{i\lambda h'(t)}\, e^{i\lambda h(t)}\Big|_a^b - \int_a^b \left(\frac{g(t)}{i\lambda h'(t)}\right)' e^{i\lambda h(t)} dt$$

$$f(\lambda) \sim \frac{g(b)}{i\lambda h'(b)} e^{i\lambda h(b)} - \frac{g(a)}{i\lambda h'(a)} e^{i\lambda h(a)} = O(\lambda^{-1}) \text{ as } \lambda \to \infty$$

<u>case 2</u> : $h'(t_0) = 0$ for some $t_0 \in (a,b)$

$$h(t) = \cancel{h(t_0)} + \cancel{h'(t_0)}(t - t_0) + \tfrac{1}{2} h''(t_0)(t-t_0)^2 + \cdots = \cancel{h(t_0)} \pm s^2$$

choose $\pm = \text{sign}(h''(t_0))$ so that $t - t_0 = \left(\dfrac{2}{|h''(t_0)|}\right)^{1/2} s + O(s^2)$

$$f(\lambda) \sim \int_{-\infty}^{\infty} g(t_0) e^{i\lambda(h(t_0) \pm s^2)} \left(\frac{2}{|h''(t_0)|}\right)^{1/2} ds = g(t_0) e^{i\lambda h(t_0)} \left(\frac{2}{|h''(t_0)|}\right)^{1/2} \int_{-\infty}^{\infty} e^{\pm i\lambda s^2} ds$$

$$= g(t_0) \left(\frac{2\pi}{\lambda |h''(t_0)|}\right)^{1/2} e^{i(\lambda h(t_0) \pm \pi/4)} = O(\lambda^{-1/2}) \text{ as } \lambda \to \infty$$

1. In summary, the leading order terms in the asymptotic expansion of $f(\lambda)$ as $\lambda \to \infty$ come from points of stationary phase and the end points of the interval.

2. These results can also be derived using the method of steepest descent; in that case a point of stationary phase corresponds to a saddle point.

<u>ex</u> : $J_n(\lambda) = \dfrac{1}{\pi} \displaystyle\int_0^\pi \cos(nt - \lambda \sin t) dt$: <u>Bessel function</u>

$$= \frac{1}{\pi} \text{Re} \int_0^\pi e^{i(nt - \lambda \sin t)} dt = \frac{1}{\pi} \text{Re} \int_0^\pi e^{-i\lambda \sin t} e^{int} dt$$

$h(t) = -\sin t\ ,\ h(t_0) = -1$

$h'(t) = -\cos t = 0 \ \Rightarrow\ t_0 = \dfrac{\pi}{2}$

$h''(t) = \sin t\ ,\ h''(t_0) = 1$

$$h(t) = h(t_0) + \cancel{h'(t_0)}(t-t_0) + \tfrac{1}{2} h''(t_0)(t-t_0)^2 + \cdots = -1 + \tfrac{1}{2}(t - \tfrac{\pi}{2})^2 + \cdots$$

$$J_n(\lambda) \sim \frac{1}{\pi} \text{Re} \int_{-\infty}^{\infty} e^{i\lambda(-1 + \frac{1}{2}t^2)} e^{in\pi/2} dt = \frac{1}{\pi} \text{Re} \left(e^{-i(\lambda - n\pi/2)} \int_{-\infty}^{\infty} e^{\frac{1}{2} i\lambda t^2} dt \right)$$

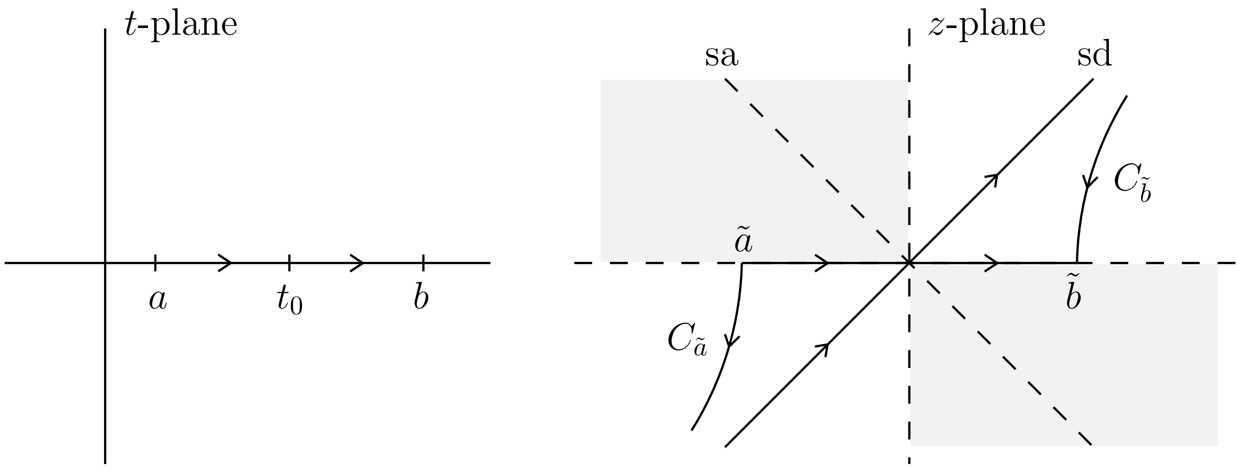
$$= \frac{1}{\pi} \text{Re} \left(e^{-i(\lambda - n\pi/2)} \left(\frac{2\pi}{\lambda}\right)^{1/2} e^{\pi i/4} \right) = \left(\frac{2}{\pi\lambda}\right)^{1/2} \cos(\lambda - n\pi/2 - \pi/4) \text{ as } \lambda \to \infty$$

<u>derivation by method of steepest descent</u>

assume $h'(t_0) = 0$, $h''(t_0) > 0$, $a < t_0 < b$

$h(t) = h(t_0) + z^2 \Rightarrow z = \pm(h(t) - h(t_0))^{1/2}$, choose $z = (\frac{1}{2}h''(t_0))^{1/2}(t - t_0) + \cdots$

$z(t_0) = 0$, $z(a) = \tilde{a} < 0$, $z(b) = \tilde{b} > 0$



$f(\lambda) = \int_a^b g(t) e^{i\lambda h(t)} dt = \int_{\tilde{a}}^{\tilde{b}} g(t(z)) e^{i\lambda h(t(z))} t'(z) dz = e^{i\lambda h(t_0)} \int_{\tilde{a}}^{\tilde{b}} g(t(z)) e^{i\lambda z^2} t'(z) dz$

apply method of steepest descent , saddle point : $z_0 = 0$

$iz^2 = i(x^2 - y^2 + 2ixy) = -2xy + i(x^2 - y^2)$

$\psi(x, y) = x^2 - y^2 = \text{cnst}$: hyperbolas , asymptotes : $y = \pm x$

$z_0 = 0 \Rightarrow \text{Im}(iz^2) = 0 \Rightarrow \psi(x, y) = x^2 - y^2 = 0 \Rightarrow y = \pm x$: sa/sd

on $y = x$, $\phi(x, y) = -2xy = -2x^2$: sd , similarly on $y = -x$, $\cdots$: sa

$C_{\tilde{a}}$, $C_{\tilde{b}}$: sd paths through $\tilde{a}$, $\tilde{b}$ $\Rightarrow \int_{\tilde{a}}^{\tilde{b}} = \int_{\text{sd}} + \int_{C_{\tilde{a}}} + \int_{C_{\tilde{b}}}$

<u>on sd</u> : $z = (1 + i)s$, $iz^2 = i(1 + i)^2 s^2 = -2s^2$

$$\int_{\text{sd}} \sim \int_{-\infty}^{\infty} g(t_0) e^{-2\lambda s^2} t'(z_0)(1 + i) ds = g(t_0) \left(\frac{2}{h''(t_0)}\right)^{1/2} \sqrt{2}\, e^{\pi i/4} \left(\frac{\pi}{2\lambda}\right)^{1/2} \text{ as } \lambda \to \infty$$

<u>on $C_{\tilde{a}}$</u> : $iz^2 = i\tilde{a}^2 - s$, $0 < s < \infty$

$$h(t) = \cancel{h(a)} + h'(a)(t - a) + O((t - a)^2) = h(t_0) + z^2 = \cancel{h(t_0) + \tilde{a}^2} + is$$

$$t = a + \frac{is}{h'(a)} + O(s^2)$$

$$\int_{C_{\tilde{a}}} \sim \int_0^{\infty} g(a) e^{i\lambda\tilde{a}^2 - \lambda s} \frac{i ds}{h'(a)} = \frac{g(a) e^{i\lambda\tilde{a}^2}}{h'(a)} \frac{i}{\lambda} \text{ as } \lambda \to 0 \text{ , similarly on } C_{\tilde{b}} \ldots$$

$$f(\lambda) \sim g(t_0) \left(\frac{2\pi}{\lambda h''(t_0)}\right)^{1/2} e^{i(\lambda h(t_0) + \pi/4)} + \frac{g(t)}{i\lambda h'(t)} e^{i\lambda h(t)} \Big|_a^b \text{ as } \lambda \to \infty \quad \underline{\text{ok}}$$

4.2 linear dispersive waves

$\phi(x,t) = \cos(kx - \omega t)$: elementary wave solution of a PDE

k : wavenumber , $k = 2\pi/L$, L : wavelength

ω : frequency , $\omega = 2\pi/T$, T : period

$kx - \omega t$: phase

If x and t vary so that $kx - \omega t = \text{cnst}$, this defines a line in xt-space on which $\phi(x,t) = \text{cnst}$; in this case the wave travels without change of shape at the phase velocity given by $v_{ph} = dx/dt = \omega/k$, which is found by substituting $\phi(x,t)$ into the PDE.

ex 1 : $\phi_t + c\phi_x = 0$: 1st order wave equation

$\phi(x,t) = \cos(kx - \omega t) \Rightarrow -\sin(kx - \omega t) \cdot -\omega + c \cdot -\sin(kx - \omega t) \cdot k = 0$

$\Rightarrow \omega - ck = 0 \Rightarrow \omega = ck \Rightarrow v_{ph} = \omega/k = c\,,\ \phi(x,t) = \cos k(x - ct)$

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ex 2 : $\phi_t + c\phi_x = \phi_{xxx}$: linearized KdV equation

$\phi(x,t) = \cos(kx - \omega t) \Rightarrow \ldots \Rightarrow \omega = ck + k^3$

In this case the phase velocity $v_{ph} = \omega/k = c + k^2$ depends on the wavenumber; hence waves with different wavenumbers travel at different speeds; this is called dispersion and $\omega = \omega(k)$ is the dispersion relation.

note : Waves of a given wavenumber travel faster in ex 2 than in ex 1.

$k \to 0 \Rightarrow v_{ph} \sim c \Rightarrow$ long waves travel at speed c

$k \to \infty \Rightarrow v_{ph} \sim k^2 \Rightarrow$ short waves travel arbitrarily fast

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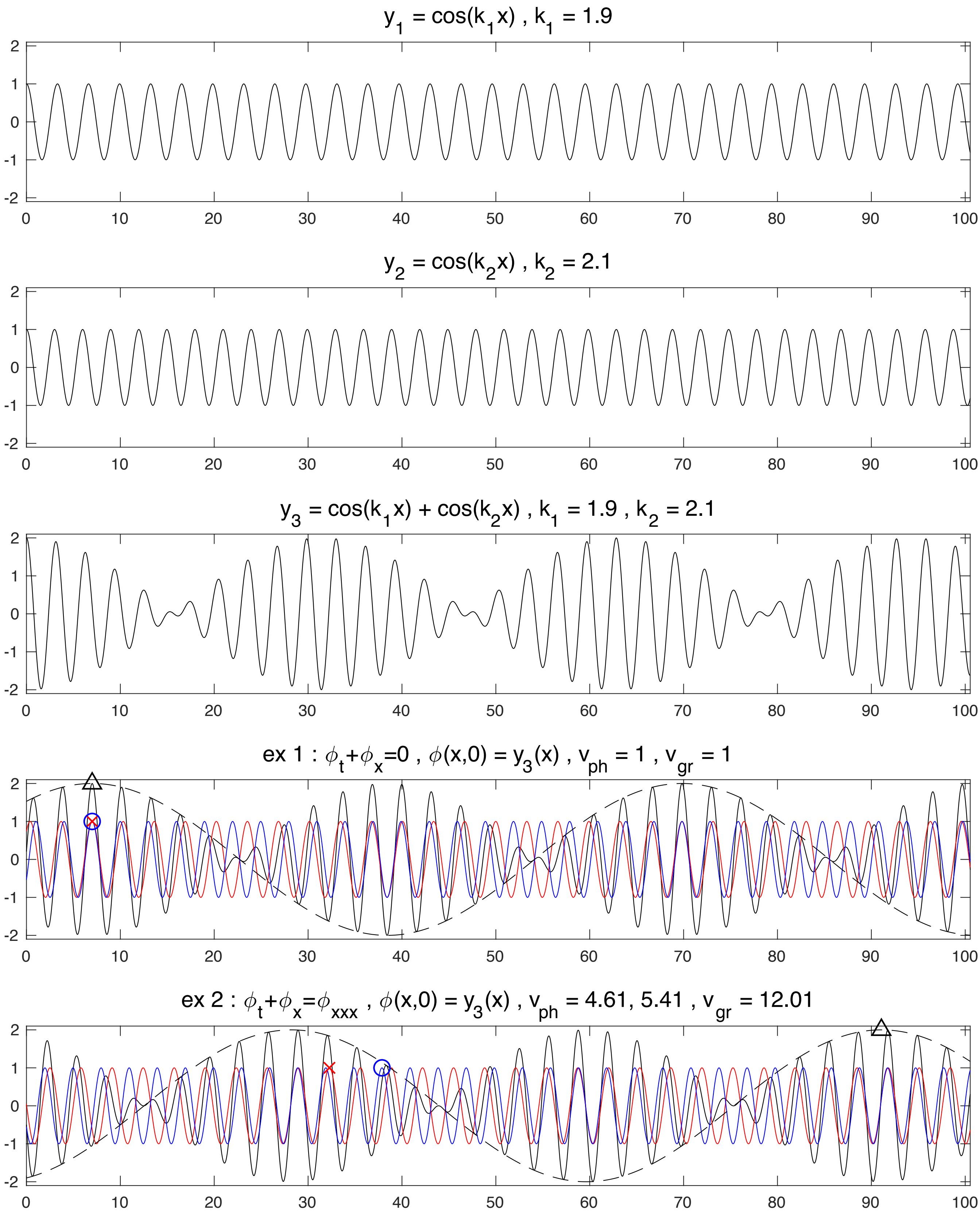
note : A superposition of two elementary waves is also a solution of the PDE; consider the case where $k_1 \approx k_2$. (Lighthill 1965, Stokes 1876)

$$\begin{aligned}\phi(x,t) &= \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)\\ &= 2\cos\tfrac{1}{2}((k_1 - k_2)x - (\omega_1 - \omega_2)t)\cos\tfrac{1}{2}((k_1 + k_2)x - (\omega_1 + \omega_2)t)\end{aligned}$$

The 1st factor on the right is a slowly varying amplitude for the rapidly varying 2nd factor; the product can be interpreted as a series of wave packets traveling at the group velocity defined by $v_{gr} = \dfrac{\omega_2 - \omega_1}{k_2 - k_1}$; for non-dispersive equations (ex 1), the phase velocity and group velocity are the same, but for dispersive equations (ex 2), they are different.

ex: $k_1 = 1.9\,,\ k_2 = 2.1\,,\ c = 1 \Rightarrow \begin{cases}\text{ex 1} : v_{ph} = 1\,,\ v_{gr} = 1\\ \text{ex 2} : v_{ph} = 4.61, 5.41\,,\ v_{gr} = 12.01\end{cases}$

$$\phi(x,t) = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)$$
$$= 2\cos\tfrac{1}{2}((k_1 - k_2)x - (\omega_1 - \omega_2)t)\cos\tfrac{1}{2}((k_1 + k_2)x - (\omega_1 + \omega_2)t)$$
$$\text{ex } 1: \ \phi_t + c\phi_x = 0 \Rightarrow \omega = ck \ \ , \ \text{ex } 2: \ \phi_t + c\phi_x = \phi_{xxx} \Rightarrow \omega = ck + k^3$$



Consider ex 1 ($\phi_t + c\phi_x = 0$) with general initial data $\phi(x,0) = f(x)$; the solution is $\phi(x,t) = f(x-ct)$ (check ...); hence the solution travels at the phase velocity c without change of shape. The solution can also be expressed as a superposition of elementary waves,

$$\phi(x,t) = \int_0^\infty a(k)\cos(kx - \omega t)dk = \int_0^\infty a(k)\cos k(x-ct)dk,$$

where the amplitude $a(k)$ is determined by the initial condition,

$$f(x) = \int_0^\infty a(k)\cos(kx)dk.$$

Consider a linear dispersive PDE with general initial data; since fast waves overtake slow waves, "an arbitrary initial disturbance disperses into a slowly varying wave train"; how can we analyze that? The solution can still be expressed as a superposition of elementary waves.

$$\phi(x,t) = \int_0^\infty a(k)\cos(kx - \omega(k)t)dk$$

$$\cos(kx - \omega(k)t) = \tfrac{1}{2}\left(e^{ith(k)} + e^{-ith(k)}\right) \ , \ h(k) = k\frac{x}{t} - \omega(k)$$

$$\phi(x,t) = \tfrac{1}{2}\int_0^\infty a(k)e^{ith(k)}dk + \tfrac{1}{2}\int_0^\infty a(k)e^{-ith(k)}dk \ : \ \text{consider } t \to \infty$$

$$h'(k) = \frac{x}{t} - \omega'(k) = 0 \Rightarrow w'(k_0) = \frac{x}{t} \ : \ \text{point of stationary phase}$$

In this context, the <u>group velocity</u> is defined by $v_{gr} = w'(k_0)$; then $x = v_{gr}t$ defines a line in xt-space on which the solution has the following asymptotic approximation.

$$\phi(x,t) \sim a(k_0)\left(\frac{2\pi}{|\omega''(k_0)|t}\right)^{1/2}\cos(k_0x - \omega(k_0)t \mp \pi/4) \text{ as } t \to \infty \ , \ \omega'(k_0) = \frac{x}{t}$$

1. $\phi(x,t) = O(t^{-1/2})$: dispersive decay , unlike ex 1

2. The asymptotic approximation resembles an elementary wave, but since k_0 depends on x/t, the coefficients are not constant; in fact they are slowly varying functions of x and t.

$$\omega'(k_0(x,t)) = \frac{x}{t} \Rightarrow w''(k_0)\cdot\frac{\partial k_0}{\partial x} = \frac{1}{t} \Rightarrow \frac{\partial k_0}{\partial x} = O(t^{-1}) \, , \, \frac{\partial k_0}{\partial t} = O(t^{-1})$$

An observer traveling at the group velocity $v_{gr} = \omega'(k_0)$ sees waves with wavenumbers near k_0; an observer standing at a fixed location $x = \omega'(k_0)t$, sees wave packets with different wavenumbers k_0 appear and disappear in time.

ex : $\phi_t + c\phi_x = \phi_{xxx}$, $\phi(x,0) = \begin{cases} 1 \ , & |x| < 1 \\ 0 \ , & |x| > 1 \end{cases}$

$\phi(x,t) = \int_0^\infty a(k)\cos(kx - \omega(k)t)dk$, $\omega(k) = ck + k^3$

$\widehat{\phi}(k,0) = \int_{-\infty}^{\infty} \phi(x,0)e^{ikx}dx$: Fourier transform

$$= \int_{-1}^{1} e^{ikx}dx = \frac{e^{ikx}}{ik}\Big|_{-1}^{1} = \frac{e^{ik} - e^{-ik}}{ik} = 2\frac{\sin k}{k}$$

$\phi(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\phi}(k,0)e^{-ikx}dk$: inverse Fourier transform

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin k}{k} e^{-ikx}dk = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin k}{k}(\cos kx - \cancel{i\sin kx})dk = \frac{2}{\pi}\int_0^\infty \frac{\sin k}{k}\cos kx dk$$

$$\phi(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\sin k}{k}\cos(kx - \omega(k)t)dk$$

recall : $h(k) = k\frac{x}{t} - \omega(k) \Rightarrow h'(k) = \frac{x}{t} - \omega'(k) = 0 \Rightarrow \omega'(k) = c + 3k^2 = \frac{x}{t}$

$\Rightarrow k_0 = \pm\left(\frac{1}{3}\left(\frac{x}{t} - c\right)\right)^{1/2}$, $\omega''(k_0) = 6k_0$

<u>case 1</u> : $x/t > c$

$\phi(x,t) \sim \frac{\sin k_0}{k_0}\left(\frac{4}{3\pi|k_0|t}\right)^{1/2}\cos(k_0x - \omega(k_0)t - \pi/4)$ as $t \to \infty$, $\omega'(k_0) = x/t$

<u>case 2</u> : $x/t = c$

$k_0 = 0$, cannot take $k_0 \to 0$ in case 1

$h(k) = -k^3 \Rightarrow h(k_0) = \max h(k) = 0$, $h'(k_0) = h''(k_0) = 0$, $h'''(k_0) = -6$

recall : hw2/#5 $\Rightarrow \phi(x,t) = O(t^{-1/3})$ as $t \to \infty$: decays slower than case 1

recall : $\phi_t + \phi_{xxx} = 0$ has a self-similar solution $\phi(x,t) = t^{-1/3}\mathrm{Ai}(-x(3t)^{-1/3})$

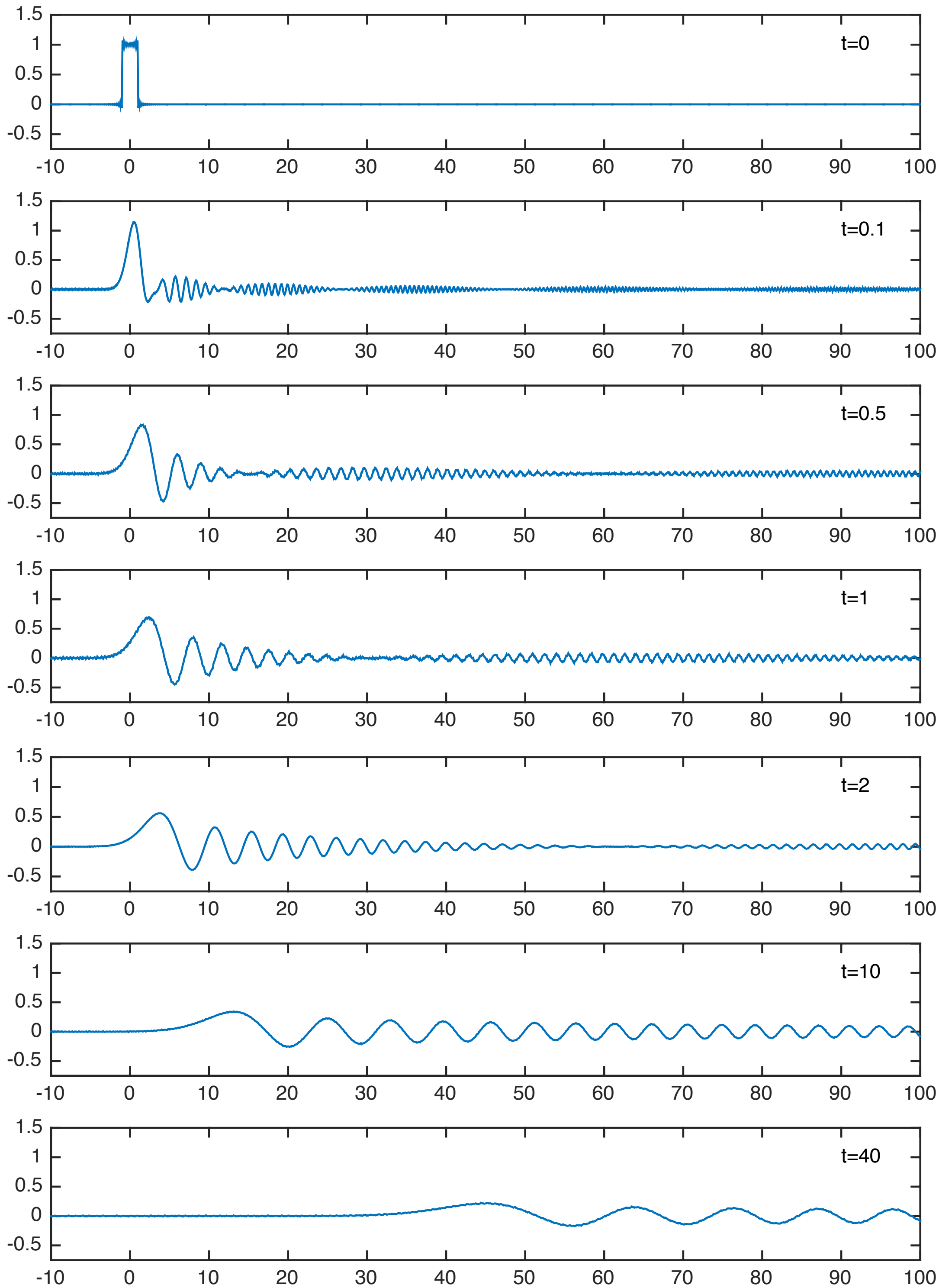
<u>case 3</u> : $x/t < c$

k_0 is imaginary $\Rightarrow$ no points of stationary phase

We can integrate by parts and show $\phi(x,t) = O(t^{-n})$ as $t \to \infty$ for all $n \geq 1$; alternatively, k_0 is a saddle point of $h(k)$ in the complex k-plane, so the method of steepest descent can be applied to show that $\phi(x,t)$ is exponentially small as $t \to \infty \ldots$

<u>ex</u> : $\phi_t + c\phi_x = \phi_{xxx}$, $c = 1$

$$\phi(x,0) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases} \Rightarrow \phi(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\sin k}{k}\cos(kx - (ck + k^3)t)dk$$



ex : $\phi_t + c\phi_x = \phi_{xxx}$, $c = 1$, case 1 : $x/t > c$, $k_0 = \pm\left(\frac{1}{3}\left(\frac{x}{t} - c\right)\right)^{1/2}$

$$\phi(x,t) \sim \frac{\sin k_0}{k_0}\left(\frac{4}{3\pi|k_0|t}\right)^{1/2} \cos(k_0 x - (ck_0 + k_0^3)t - \pi/4) \text{ as } t \to \infty$$